

ON THE BASED RING ATTACHED TO THE SUBREGULAR CELL OF A COXETER GROUP

TIANYUAN XU

ABSTRACT. Let (W, S) be an arbitrary Coxeter system, and let J be the asymptotic Hecke algebra associated to (W, S) via Kazhdan-Lusztig polynomials by Lusztig. We study a subalgebra J_C of J corresponding to the subregular cell C of W and prove a factorization theorem that allows us to compute products in J_C without inputs from Kazhdan-Lusztig theory. We discuss two applications of this result. First, we describe J_C in terms of the Coxeter diagram of (W, S) in the case (W, S) is simply-laced, and deduce more connections between the diagram and J_C in some other cases. Second, we prove that for certain specific Coxeter systems, some subalgebras of J_C are free fusion rings in the sense of [BV09], thereby connecting the algebras to compact quantum groups arising in operator algebra theory.

1. INTRODUCTION

Hecke algebras of Coxeter systems are classical objects of study in representation theory because of their rich connections with finite groups of Lie type, Lie algebras, quantum groups, and the geometry of flag varieties (see, for example, [Cur87], [CIK71], [DJ86], [GP00], [KL79], [Lus84]). Let (W, S) be a Coxeter system, and let H be its Hecke algebra defined over the ring $\mathbb{Z}[v, v^{-1}]$. Using Kazhdan-Lusztig polynomials, Lusztig constructed the *asymptotic Hecke algebra* J of (W, S) from H in [Lus87a]. The algebra J can be viewed as a limit of H as the parameter v goes to infinity, and its representation theory is closely related to that of H (see [Lus87a], [Lus87b], [Lus89], [Lus14], [Gec98]). In particular, upon suitable extensions of scalars, J admits a natural homomorphism from H , hence representations of J induce representations of H ([Lus14]).

The asymptotic Hecke algebra J has several interesting features. First, given a Coxeter system (W, S) , J is defined to be the free abelian group $J = \oplus_{w \in W} \mathbb{Z}t_w$, with multiplication of the basis elements declared by

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z$$

where the coefficients $\gamma_{x,y,z^{-1}}$ ($x, y, z \in W$) are nonnegative integers extracted from the structure constants of the *Kazhdan-Lusztig basis* of the Hecke algebra H of (W, S) . The non-negativity of its structure constants makes J a \mathbb{Z}_+ -ring, and the basis elements satisfy additional conditions which make J a *based ring* in the sense of [Lus87c] and [EGNO15] (see §3.3).

Another interesting feature of J is that for any *2-sided Kazhdan-Lusztig cell* E of W , the subgroup

$$J_E = \oplus_{w \in E} \mathbb{Z}t_w$$

of J is a subalgebra of J and also a based ring. Here, as the notation suggests, a 2-sided Kazhdan-Lusztig cell is a subset of W . The cells of W are defined using the Kazhdan-Lusztig basis of its associated Hecke algebra H and form a partition of W . Further, the subalgebra J_E is in fact a direct summand of J for each 2-sided

cell E , and J admits the direct sum decomposition

$$J = \oplus_{E \in \mathcal{C}} J_E,$$

where \mathcal{C} denotes the collection of all 2-sided cells of W (see §3.2). Thus, it is natural to study J by first studying its direct summands corresponding to the cells.

In this paper, we focus on a particular 2-sided cell C of W known as the *subregular cell* and study the titular based ring J_C . We also study subalgebras J_s of J_C that correspond to the generators $s \in S$ of W . Thanks to a result of Lusztig in [Lus83], the cell C can be characterized as the set of elements in W with unique reduced expressions, and the main theme of the paper is to exploit this combinatorial characterization and study J_C and $J_s (s \in S)$ without reference to Kazhdan-Lusztig polynomials. This is desirable since a main obstacle in understanding J for arbitrary Coxeter systems lies in the difficulty of understanding Kazhdan-Lusztig polynomials.

A third important feature of the algebra J is that it has very interesting *categorification*. Here by categorification we mean the process of adding an extra layer of structure to an algebraic object to produce an interesting category which allows one to recover the object; more specially, we mean J appears as the Grothendieck ring of a *tensor category* \mathcal{J} (see [EGNO15] for the definition of a tensor category, [Lus14] for the construction of \mathcal{J}). A well-known example of categorification is the categorification of the Hecke algebra H by the *Soergel category* \mathcal{SB} , which was used to prove the “positivity properties” of the *Kazhdan-Lusztig basis* of H in [EW14].

Just as the algebra J is constructed from H , the category \mathcal{J} is constructed from the category \mathcal{SB} , also by Lusztig ([Lus14]). Further, just as the algebra J has a subalgebra of the form J_E for each 2-sided cell E and a subalgebra J_s for each generator $s \in S$, the category \mathcal{J} has a subcategory \mathcal{J}_E for each 2-sided cell E and a subcategory \mathcal{J}_s for each $s \in S$. Moreover, \mathcal{J}_E categorifies J_E for each 2-sided cell E , and \mathcal{J}_E is a *multifusion category* in the sense of [EGNO15] whenever E is finite, which can happen for suitable cells even when the ambient group W is infinite. Similarly, \mathcal{J}_s is a *fusion category* whenever J_s has finite rank. Multifusion and fusion categories have rich connections with quantum groups ([Kas95]), conformal field theory ([MS89]), quantum knot invariants ([Tur10]) and topological quantum field theory ([BK01]), so the categories \mathcal{J}_E (in particular, \mathcal{J}_C) and \mathcal{J}_s are interesting since they can potentially provide new examples of multifusion and fusion categories.

Historically, the intimate connection between the algebra J and its categorification \mathcal{J} has been a major tool in the study of both objects. For Weyl groups and an affine Weyl groups, Lusztig ([Lus89], [Lus97]) and Bezrukanikov et al. ([Bez04], [BO04], [BFO09]) showed that there is a bijection between the two-sided cells in the group and unipotent conjugacy classes of an algebraic group, and that the subcategories of \mathcal{J} corresponding to the cells can be described geometrically, as categories of vector bundles on a square of a finite set equivariant with respect to an algebraic group. Using the categorical results, they computed the structure constants in J explicitly. For other Coxeter systems, however, the nature of J or \mathcal{J} seems largely unknown, partly because there is no known recourse to advanced geometry. In this context, our paper may be viewed as an attempt to understand the subalgebra J_C of J for arbitrary Coxeter systems from a more combinatorial point of view. We hope to understand the structure of J_C by examining the multiplication rule in J_C , then, in some cases, use our knowledge of J to deduce the structure of \mathcal{J} . This idea is further discussed in §6.1.

The main results of the paper fall into two sets. First, we describe some connections between the *Coxeter diagram* G of an arbitrary Coxeter system (W, S) and the algebra J_C associated to (W, S) . The first result in this spirit describes J_C in

terms of G for all *simply-laced* Coxeter systems. Recall that given any vertex s in G , the fundamental group $\Pi_s(G)$ of G based at s is the group consisting of all homotopy equivalence classes of walks in G starting and ending at s , equipped with concatenation as the group operation. One may generalize this notion to define the *fundamental groupoid* $\Pi(G)$ of G as the set of homotopy equivalence classes of all walks on G , equipped with concatenation as a partial binary operation that is defined between two classes when their concatenation makes sense. We define the groupoid algebra of $\mathbb{Z}\Pi(G)$ of $\Pi(G)$ by mimicking the construction of a group algebra from a group, and we prove the following theorem.

Theorem A. *Let (W, S) be an any simply-laced Coxeter system, and let G be its Coxeter diagram. Let $\Pi(G)$ be the fundamental groupoid of G , let $\Pi_s(G)$ be the fundamental group of G based at s for any $s \in S$, let $\mathbb{Z}\Pi(G)$ be the groupoid algebra of $\Pi(G)$, and let $\mathbb{Z}\Pi_s(G)$ be the group algebra of $\Pi_s(G)$. Then $J_C \cong \mathbb{Z}\Pi(G)$ as based rings, and $J_s \cong \mathbb{Z}\Pi_s(G)$ as based rings for all $s \in S$.*

The key idea behind the theorem is to find a correspondence between basis elements of J_C and classes of walks on G . The correspondence then yields explicit formulas for the claimed isomorphisms.

In our second result, we study the case where G is *oddly-connected*. Here by oddly-connected we mean that each pair of distinct vertices in G are connected by a path involving only edges of odd weights.

Theorem B. *Let (W, S) be an oddly-connected Coxeter system. Then*

- (1) $J_s \cong J_t$ as based rings for all $s, t \in S$.
- (2) $J_C \cong \text{Mat}_{S \times S}(J_s)$ as based rings for all $s \in S$. In particular, J_C is Morita equivalent to J_s for all $s \in S$.

Once again, we will provide explicit isomorphisms between the algebras using G .

In a third result, we describe all *fusion rings* that appear in the form J_s for some Coxeter system (W, S) and some choice of $s \in S$. We show that any such fusion ring is isomorphic to a ring J_s associated to a dihedral system, which is in turn always isomorphic to the *odd part* of a *Verlinde algebra* associated to the Lie group $SU(2)$ (see Definition 4.19).

Theorem C. *Let (W, S) be a Coxeter system, and let $s \in S$. Suppose J_s is a fusion ring for some $s \in S$. Then there exists a dihedral Coxeter system (W', S') such that $J_s \cong J_{s'}$ as based rings for either $s' \in S$.*

In our second set of results, we focus on certain specific Coxeter systems (W, S) whose Coxeter diagram involves edges of weight ∞ , and show that for suitable choices of $s \in S$, J_s is isomorphic to a *free fusion ring* in the sense of [BV09]. A free fusion ring can be described in terms of the data of its underlying *fusion set*, and we describe these data explicitly for each free fusion ring J_s in our examples. Furthermore, each free fusion ring we discuss is isomorphic to the Grothendieck rings of the category $\text{Rep}(\mathbb{G})$ of representations of a known *partition quantum groups* \mathbb{G} , and we will identify the group \mathbb{G} in all cases. Our main theorems appear as Theorem D and Theorem E in §6.3 and §6.4, but we omit their technical statements for the moment.

All the results mentioned above rely heavily on the following theorem, which says that a combinatorial “factorization” of a reduced word of an element into its *dihedral segments* (see Definition 4.2) carries over to a factorization of basis elements in J_C .

Theorem F. (Dihedral factorization) *Let x be the reduced word of an element in C , and let x_1, x_2, \dots, x_l be the dihedral segments of x . Then*

$$t_x = t_{x_1} \cdot t_{x_2} \cdot \dots \cdot t_{x_l}.$$

The rest of the article is organized as follows. We review some preliminaries about Coxeter systems and Hecke algebras in Section 2. In Section 3, we define the algebras J , J_C and $J_s (s \in S)$ and explain how J_C and $J_s (s \in S)$ appear as based rings. We prove Theorem F in Section 4 and show how it can be used to compute products of basis elements in J_C . In Section 5, we prove our results on the connections between J_C and Coxeter diagrams. Finally, we discuss our second set of results in Section 6, where we prove that certain rings J_s are free fusion rings.

Acknowledgements. I would like to thank Victor Ostrik for his guidance and numerous helpful suggestions. I am also very grateful to Alexandru Chirvasitu and Amaury Freslon for helpful discussions about free fusion rings and compact quantum groups. Finally, I would like to acknowledge the mathematical software SageMath ([Dev16]), which was used extensively for many of our computations.

2. PRELIMINARIES

In this section we review the basic theory of Coxeter systems and Hecke algebras relevant to this paper. Our main references are [BB05] and [Lus14]. In particular, we define the Hecke algebras over the ring $\mathbb{Z}[v, v^{-1}]$ and use a normalization seen in [Lus14], where the quadratic relations are $(T_s - v)(T_s + v^{-1}) = 0$ for all simple reflections s . Otherwise the treatment is standard and self-contained. Readers familiar with these topics may skip this section entirely.

2.1. Coxeter systems. A *Coxeter system* is a pair (W, S) where S is a set equipped with a map $m : S \times S \rightarrow \mathbb{Z}_{\geq 1} \cup \{\infty\}$ such that $m(s, s) = 1$ and $m_{s,t} = m_{t,s} \geq 2$ for all distinct elements $s, t \in S$, and W is the group presented by

$$W = \langle S \mid (st)^{m(s,t)} = 1, \forall s, t \in S \rangle.$$

The group W is called a *Coxeter group*. If W is a finite group, we say (W, S) is a *finite Coxeter system*.

Example 2.1. (1) (Dihedral groups) Let $n \in \mathbb{Z}_{\geq 3}$ and let (W, S) be the Coxeter system with $S = \{s, t\}$ and $W = \langle s, t \mid s^2 = t^2 = (st)^n = 1 \rangle$. Then W is isomorphic to the dihedral group D_n of order $2n$, the group of symmetries of a regular n -gon P . To see this, let c be the center of P , let d a vertex of P , and let e be the midpoint of an edge incident to d . Let s' and t' be reflections with respect to the two lines going through c, d and through c, e , respectively. Then s', t' are involutions since they are reflections. Since the two lines form an angle of π/n , $s't'$ is rotation at an angle of $2\pi/n$, hence $(s't')^n = 1$. It follows that the map $s \mapsto s', t \mapsto t'$ extends uniquely to a group homomorphism $\varphi : W \rightarrow D_n$. The map is surjective since s', t' generate D_n , and a moment's thought reveals that $|W| \leq 2n = |D_n|$, therefore φ must be an isomorphism.

(2) (Symmetric groups) Let $n \in \mathbb{Z}_{\geq 2}$, $S = \{s_1, s_2, \dots, s_{n-1}\}$, and let W be the Coxeter group generated by S subject to the relations $s_i^2 = 1$ for all i , $(s_i s_j)^3 = 1$ for all i, j with $|i - j| = 1$, and $(s_i s_j)^2 = 1$ for all i, j with $|i - j| > 1$. Then W is isomorphic to the symmetric group S_n . More precisely, let s'_i be the i -th basic transposition $(i, i + 1)$ in S_n , then it is straightforward to check that the map $s_i \mapsto s'_i$ extends to a group isomorphism from W to S_n .

(3) (Weyl groups) The Weyl group of a *root system* ([Hum90]) is a Coxeter group. Weyl groups constitute the majority of finite Coxeter groups (see [BB05]).

The data of a Coxeter system (W, S) can be efficiently encoded via a *Coxeter diagram* G . By definition, G is the loopless, weighted, undirected graph (V, E) with vertex set $V = S$ and with edges E given as follows. For any distinct $s, t \in S$, $\{s, t\}$ forms an edge in G exactly when $m(s, t) \geq 3$, whence the weight of the edge is $m(s, t)$. When drawing a Coxeter graph, it is conventional to leave edges of weight

3 unlabeled. We call edges of weight 3 *simple*, and we say (W, S) is *simply-laced* if all edges of G are simple.

We call a Coxeter system (W, S) *irreducible* if its Coxeter graph G is connected. This terminology comes from the following fact. If G is not connected, then each connected component of G encodes a Coxeter system. Since $m(s, t) = 2$ for any vertices s, t selected from different connected components of G , and since $s^2 = t^2 = 1$ now that $m(s, s) = m(t, t) = 1$, we have $st = ts$. This means that the Coxeter groups corresponding to the components commute with each other, so W is isomorphic to the direct product of these Coxeter groups and hence “reducible”. That said, for most purposes we may study only irreducible Coxeter systems.

2.2. Combinatorics of words. Let (W, S) be a Coxeter system, and let $\langle S \rangle$ be the free monoid generated by S . It is natural to think of elements in W as represented by elements of $\langle S \rangle$, or *words* or *expressions* in the alphabet S . We review some basic facts about words in Coxeter groups in this subsection.

For $w \in W$, we define the *length* of w in W , written $l(w)$, to be the minimal length of a word representing w , and we call any such minimal-length word a *reduced word* or *reduced expression* of w . As we shall see, reduced words lie at the heart of the combinatorics of Coxeter groups.

Our first fact concerns the order of products of the form st ($s, t \in S$) in W . Recall that W is presented by

$$(1) \quad W = \langle S \mid (st)^{m(s,t)} = 1, \forall s, t \in S \rangle,$$

hence the order of the element st divides $m(s, t)$ for any $s, t \in S$. But more is true:

Proposition 2.2 ([Lus14], Proposition 1.3). *If $s \neq t$ in S , then st has order $m(s, t)$ in W . In particular, $s \neq t$ in W .*

In light of the proposition, we shall henceforth identify S with a subset of W and call S the set of *simple reflections* in W . For $s, t \in S$, since $s^2 = t^2 = 1$, the relation $(st)^{m(s,t)} = 1$ is equivalent to

$$(2) \quad sts \cdots = tst \cdots,$$

where both sides are words that alternate in s and t and have length $m(s, t)$. Such a relation is called a *braid relation*.

Example 2.3. Let W be the dihedral group with Coxeter generators $S = \{1, 2\}$ and $m(1, 2) = M$ for some $M \geq 3$. For $0 \leq k \leq M$, let 1_k and 2_k be the alternating words $121 \cdots$ and $212 \cdots$ of length k , respectively. In particular, set $1_0 = 2_0 = 1_W$, the identity element of W . By Proposition 2.2 and the braid relations, if $M < \infty$, then W consists of the $2M$ elements $1_k, 2_k$ where $0 \leq k \leq M$, and they are all distinct except the equalities $1_0 = 2_0$ and $1_M = 2_M$; if $M = \infty$, then W consists of the elements $1_k, 2_k$ for all $k \in \mathbb{Z}_{\geq 0}$, and they are all distinct except for $1_0 = 2_0$. Moreover, it is clear that $l(1_k) = l(2_k) = k$ for all $0 \leq k \leq M$.

Our second fact concerns the reduced expressions of a fixed element in W . Note that the braid relations mean that whenever one side of Equation (2) appears consecutively in a word representing an element in W , we may replace it with the other side of the equation and obtain a different expression of the same element. Call such a move a *braid move*. Then we have:

Proposition 2.4 (Matsumoto’s Theorem; see, e.g., [Lus14], Theorem 1.9). *Any two reduced words of a same element in W can be obtained from each other by performing a finite sequence of braid moves.*

Our third fact concerns the *descent sets* of elements in W . For $x \in W$, define the *left descent set* and *right descent set* of x to be the sets

$$\mathcal{L}(x) = \{s \in S : l(sx) < l(x)\},$$

$$\mathcal{R}(x) = \{s \in S : l(xs) > l(x)\},$$

respectively. Descent sets can again be characterized in terms of reduced words:

Proposition 2.5 (Descent criterion; [BB05], Corollary 1.4.6). *Let $s \in S$ and $x \in W$. Then*

- (1) $s \in \mathcal{L}(x)$ if and only if x has a reduced word beginning with s ;
- (2) $s \in \mathcal{R}(x)$ if and only if x has a reduced word ending with s .

Finally, our fourth fact concerns the *Bruhat order* \leq on W . The Bruhat order is an important partial order on W defined as follows. First, define a *reflection* in W to be a conjugate of any simple reflection, i.e., an element of the form $t = s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_2 s_1$ where $s_i \in S$ for all $1 \leq i \leq k$. Then, declare a relation \prec on W such that $x \prec y$ for $x, y \in W$ if and only if $x = ty$ and $l(x) < l(y)$ for some reflection t . Finally, take the Bruhat order \leq to be the reflexive and transitive closure of \prec .

Once again, our fact says that there is a characterization of the Bruhat order in terms of reduced words. Define a *subword* of any word $s_1 s_2 \cdots s_k \in S^*$ to be a word of the form $s_{i_1} s_{i_2} \cdots s_{i_l}$ where $1 \leq i_1 < i_2 < \cdots < i_l \leq k$. The fact says:

Proposition 2.6 (Subword Property; [BB05], Corollary 2.2.3). *Let $x, y \in W$. Then the following are equivalent:*

- (1) $x \leq y$;
- (2) every reduced word for y contains a subword that is a reduced word for x ;
- (3) some reduced word for y contains a subword that is a reduced word for x .

This immediately implies the following:

Corollary 2.7 ([BB05], Corollary 2.2.5). *The map $w \mapsto w^{-1}$ on W is an automorphism of the Bruhat order, i.e., $u \leq w$ if and only if $u^{-1} \leq w^{-1}$.*

2.3. Hecke algebras. In this subsection we review some basic facts about Hecke algebras and their Kazhdan-Lusztig theory. Throughout, let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$.

Let (W, S) be an arbitrary Coxeter system. Following [Lus14], we define the *Iwahori-Hecke algebra* (or simply the *Hecke algebra* for short) of (W, S) to be the unital \mathcal{A} -algebra H generated by the set $\{T_s : s \in S\}$ subject to the relations

$$(3) \quad (T_s - v)(T_s + v^{-1}) = 0$$

for all $s \in S$ and the relations

$$(4) \quad T_s T_t T_s \cdots = T_t T_s T_t \cdots$$

for all $s, t \in S$, where both sides have $m(s, t)$ factors. Note that when we set $v = 1$, the quadratic relation reduces to $T_s^2 = 1$, so H is isomorphic to the group algebra $\mathbb{Z}W$ of W by the braid relations in W and (4). In this sense, H is often called a *deformation* of $\mathbb{Z}W$.

Let $x \in W$, let $s_1 s_2 \cdots s_k$ be any reduced word of x , and set $T_x := T_{s_1} \cdots T_{s_k}$. Thanks to Proposition 2.4 and Equation (4), this is well-defined, i.e., different reduced words of x produce the same element in H . The following is well known.

Proposition 2.8 ([Lus14], Proposition 3.3). *The set $\{T_x : x \in W\}$ is an \mathcal{A} -basis of H .*

The basis $\{T_x : x \in W\}$ is called the *standard basis* of H . In the seminal paper [KL79], Kazhdan and Lusztig introduced another basis of $\{c_x : x \in W\}$ of H that is now known as the *Kazhdan-Lusztig basis*. The transition matrices between the two bases give rise to the famous *Kazhdan-Lusztig polynomials*. By definition, they are the elements $p_{x,y} \in \mathcal{A}$ for which

$$c_y = \sum_{x \in W} p_{x,y} T_x$$

for all $x, y \in W$.

Notation 2.9. From now on we will mention the phrase “Kazhdan-Lusztig” numerous times. We will often abbreviate it to “KL”.

Remark 2.10. In the paper [KL79] where the KL basis and polynomials were first introduced, the Hecke algebra H is defined over a ring $\mathbb{Z}[q]$ and the KL polynomials are polynomials in q . Under our choice of normalization for H , however, we actually have $p_{x,y} \in \mathbb{Z}[v^{-1}]$ for all $x, y \in W$ (see [Lus14], §5).

2.4. Kazhdan-Lusztig theory. KL bases and KL polynomials are essential to the construction of asymptotic Hecke algebras. We recall the relevant facts below.

First, for $x, y, z \in W$, let $h_{x,y,z}$ be the unique elements in \mathcal{A} such that

$$(5) \quad c_x c_y = \sum_{z \in W} h_{x,y,z} c_z.$$

The following theorem says that both the KL polynomials $p_{x,y}$ and the coefficients $h_{x,y,z}$ always have nonnegative integer coefficients.

Theorem 2.11 (Positivity of the KL basis and KL polynomials; [EW14]).

- (1) $p_{x,y} \in \mathbb{N}[v, v^{-1}]$ for all $x, y \in W$.
- (2) $h_{x,y,z} \in \mathbb{N}[v, v^{-1}]$ for all $x, y, z \in W$.

As mentioned in the introduction, these facts are proved using the categorification of H by the Soergel category \mathcal{SB} associated with the Coxeter system of H .

Remark 2.12. It is well-known that KL polynomials can be computed recursively with the aid of the so-called *R-polynomials*. This is explained in sections 4 and 5 of [Lus14], and example computations can be found in Chapter 5 of [BB05]. However, the computation is often very difficult to carry out in practice, even for computers, and the computation algorithm does not seem adequate for a proof of part (1) of the above theorem.

Second, we recall a multiplication formula for KL basis elements in H . For $x, y \in W$, let $\mu_{x,y}$ denote the coefficient of v^{-1} in $p_{x,y}$ and call it a μ -coefficient. The μ -coefficients can be used to define representations of H via *W-graphs* ([KL79]). They also govern the multiplication of KL basis elements in H :

Proposition 2.13 (Multiplication of KL-basis; [Lus14], Theorem 6.6, Corollary 6.7). *Let $x \in W$, $s \in S$, and let \leq be the Bruhat order on W . Then*

$$\begin{aligned} c_s c_y &= \begin{cases} (v + v^{-1})c_y & \text{if } sy < y \\ c_{sy} + \sum_{z: sx < x < y} \mu_{x,y} c_z & \text{if } sy > y \end{cases}, \\ c_y c_s &= \begin{cases} (v + v^{-1})c_y & \text{if } ys < y \\ c_{ys} + \sum_{x: xs < x < y} \mu_{x^{-1}, y^{-1}} c_x & \text{if } sy > y \end{cases}. \end{aligned}$$

Next, we define *Kazhdan-Lusztig cells*. For each $x \in W$, let $D_x : H \rightarrow \mathcal{A}$ be the linear map such that

$$D_x(c_y) = \delta_{x,y}$$

for all $y \in W$, where $\delta_{x,y}$ is the Kronecker delta symbol. For $x, y \in W$, write $x \prec_L y$ if $D_x(c_s c_y) \neq 0$ for some $s \in S$, and write $x \prec_R y$ if $D_x(c_y c_s) \neq 0$ for some $s \in S$. Define \leq_L and \leq_R to be the transitive closures of \prec_L and \prec_R , respectively, and define another partial order \leq_{LR} by declaring that $x \leq_{LR} y$ if there exists a sequence $x = z_1, \dots, z_n = y$ in W such that $z_i \prec_L z_{i+1}$ or $z_i \prec_R z_{i+1}$ for all $1 \leq i \leq n-1$. Finally, define \sim_L to be the equivalence relations such that $x \sim_L y$ if and only if we have both $x \leq_L y$ and $y \leq_L x$, and define \sim_R, \sim_{LR} similarly. The equivalence classes of \sim_L, \sim_R and \sim_{LR} are called the *left (Kazhdan-Lusztig) cells*, *right (Kazhdan-Lusztig) cells* and *2-sided (Kazhdan-Lusztig) cells* of W , respectively. Clearly, each 2-sided KL cell is a union of left cells as well as a union of right cells. Since the elements c_s ($s \in S$) generate H as an \mathcal{A} -algebra, the following is also clear:

Proposition 2.14 ([Lus14], Lemma 8.2). *Let $y \in W$. Then*

- (1) *The set $H_{\leq_L y} := \oplus_{x: x \leq_L y} \mathcal{A}c_x$ is a left ideal of H .*
- (2) *The set $H_{\leq_R y} := \oplus_{x: x \leq_R y} \mathcal{A}c_x$ is a right ideal of H .*
- (3) *The set $H_{\leq_{LR} y} := \oplus_{x: x \leq_{LR} y} \mathcal{A}c_x$ is a 2-sided ideal of H .*

Observe that by Proposition 2.13, we have $x \prec_L y$ if and only if either $x = y$ or $x < y, \mu_{x,y} \neq 0$ and $\mathcal{L}(x) \not\subseteq \mathcal{L}(y)$. Similarly, $x \prec_R y$ if and only if either $x = y$ or $x < y, \mu_{x^{-1}, y^{-1}} \neq 0$ and $\mathcal{R}(x) \not\subseteq \mathcal{R}(y)$. Corollary 2.7 now implies that $x \leq_L y$ if and only if $x^{-1} \leq_R y^{-1}$. We have just proved

Proposition 2.15 ([Lus14], Section 8.1). *The map $x \mapsto x^{-1}$ takes left cells in W to right cells, right cells to left cells, and 2-sided cells to 2-sided cells.*

We will need to use one more fact later.

Proposition 2.16 ([Lus14], Section 14.1). *For any $x \in W$, we have $x \sim_{LR} x^{-1}$.*

3. THE SUBREGULAR J -RING

In this section, we describe Lusztig's construction of the asymptotic Hecke algebra J of a Coxeter system and recall some basic properties of J . We show how KL cells in W give rise to subalgebras of J , then shift our focus to a particular algebra J_C of J corresponding to the subregular cell of W . We also recall the definition of a based ring and explain why J_C is a based ring.

Throughout the section, suppose (W, S) is an arbitrary Coxeter system with $S = [n] = \{1, 2, \dots, n\}$ unless otherwise stated. Let H be the Iwahori-Hecke algebra of (W, S) , and let $\{T_w : w \in W\}, \{c_w : w \in W\}$ and $\{p_{y,w} : y, w \in W\}$ be the standard basis, KL basis and KL polynomials in H , respectively.

3.1. The asymptotic Hecke algebra J . Consider the elements $h_{x,y,z} \in \mathbb{Z}[v, v^{-1}]$ ($x, y, z \in W$) from Equation 5. Lusztig showed in [Lus14] that for any $z \in W$, there exists a unique integer $\mathbf{a}(z) \geq 0$ that satisfies the conditions

- (a) $h_{x,y,z} \in v^{\mathbf{a}(z)} \mathbb{Z}[v^{-1}]$ for all $x, y \in W$,
- (b) $h_{x,y,z} \notin v^{\mathbf{a}(z)-1} \mathbb{Z}[v^{-1}]$ for some $x, y \in W$.

Define $\gamma_{x,y,z^{-1}}$ to be the non-negative integer such that

$$h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{\mathbf{a}(z)} \pmod{v^{\mathbf{a}(z)-1} \mathbb{Z}[v^{-1}]},$$

and define multiplication on the free abelian group $J = \oplus_{w \in W} \mathbb{Z}w$ by

$$t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z$$

for all $x, y \in W$. It is known in that this product is well-defined (i.e., $\gamma_{x,y,z^{-1}} = 0$ for all but finitely many $z \in W$ for all $x, y \in W$), and the multiplication defined above is associative, making J a ring (see [Lus14], 18.3). We call J the *asymptotic Hecke algebra* or simply the *J -ring* of (W, S) .

The following facts about the coefficients $\gamma_{x,y,z}$ and J will be useful later.

Proposition 3.1 ([Lus14], Proposition 13.9). *For all $x, y, z \in W$, $\gamma_{y^{-1},x^{-1},z^{-1}} = \gamma_{x,y,z}$.*

Note that this immediately implies the following.

Corollary 3.2. *The \mathbb{Z} -linear map with $t_x \mapsto t_{x^{-1}}$ is an anti-homomorphism of J .*

3.2. Subalgebras of J . For each $x \in W$, let $\Delta(x)$ be the unique non-negative integer such that

$$p_{1,x} \in n_x v^{-\Delta(x)} + v^{-\Delta(x)-1} \mathbb{Z}[v^{-1}]$$

for some $n_x \neq 0$. This makes sense by Remark 2.12. Let

$$\mathcal{D} = \{x \in W : \mathbf{a}(x) = \Delta(x)\}.$$

It is known that $d^2 = 1$ for all $d \in \mathcal{D}$, and \mathcal{D} is called the set of *distinguished involutions*. There are many intricate connections between \mathcal{D} , the coefficients $\gamma_{x,y,z}$, and KL cells in W . The connections would lead us to many subalgebras of J that are indexed by cells and have units provided by the distinguished involutions.

Proposition 3.3 ([Lus14], Conjectures 14.2). *Let $x, y, z \in W$. Then*

- (1) $\gamma_{x,y,z} = \gamma_{y,z,x}$.
- (2) If $\gamma_{x,y,z} \neq 0$, then $x \sim_L y^{-1}, y \sim_L z^{-1}, z \sim_L x^{-1}$.
- (3) If $\gamma_{x,y,d} \neq 0$ for some $d \in \mathcal{D}$, then $y = x^{-1}$ and $\gamma_{x,y,d} = 1$. Further, for each $x \in W$ there is a unique element $d \in \mathcal{D}$ such that $\gamma_{x,x^{-1},d} = 1$.
- (4) Each left KL cell Γ of W contains a unique element d from \mathcal{D} . Further, for this elements d , we have $\gamma_{x^{-1},x,d} = 1$ for all $x \in \Gamma$.

Remark 3.4. In the paper [Lus87a], where Lusztig first defined the asymptotic Hecke algebra J , Proposition 3.3 is proved for Coxeter systems satisfying certain mild conditions. The conditions can be found in Section 1.1 of the paper, the four parts of the proposition appear in Theorem 1.8, Corollary 1.9, Proposition 1.4 and Theorem 1.10 of the paper, respectively. For arbitrary Coxeter systems, the statements of the proposition, as well as the statement in Proposition 2.16, appear only as conjectures in Chapter 14 of [Lus14]. However, [Lus14] studies Hecke algebras in a more general setting, namely, with possibly *unequal parameters*, and the statements are known to be true in the setting of this paper, which is called the *equal parameter* or the *split* case in the book. The proofs of the statements rely heavily on Theorem 2.11; see Chapter 15 of [Lus14].

Definition 3.5. For any subset X of W , define $J_X := \oplus_{w \in X} \mathbb{Z} t_w$.

Corollary 3.6 ([Lus14], Section 18.3).

- (a) Let Γ be any left KL cell in W , say with $\Gamma \cap \mathcal{D} = \{d\}$. Then the subgroup $J_{\Gamma \cap \Gamma^{-1}}$ is actually a unital subalgebra of J ; its unit is t_d .
- (b) For any 2-sided cell E in W , the subgroup J_E is a subalgebra of J . Further, we have a direct sum decomposition $J = \oplus_{E \in \mathcal{C}} J_E$ of algebras, where \mathcal{C} is the collection of all 2-sided KL cells of W .
- (c) If E is a 2-sided cell such that $E \cap \mathcal{D}$ is finite, then J_E is a unital algebra with unit element $\sum_{d \in E \cap \mathcal{D}} t_d$.
- (d) If \mathcal{D} is finite, then J is a unital algebra with unit $\sum_{d \in \mathcal{D}} t_d$.

Proof. We will repeatedly use Proposition 3.3. When we say part (i), we will mean part (i) of the proposition.

- (a) Let $x, y \in \Gamma \cap \Gamma^{-1}$, and suppose $\gamma_{x,y,z^{-1}} \neq 0$ for some $z \in W$. Then by part (2), $z = (z^{-1})^{-1} \sim_L y \in \Gamma$, and $z^{-1} \sim_L x^{-1}$ so that $z \sim_R x \in \Gamma^{-1}$ (since the inverse map takes left cells to right cells by Proposition 2.15). Thus, $z \in \Gamma \cap \Gamma^{-1}$. It follows that $J_{\Gamma \cap \Gamma^{-1}}$ is a subalgebra of J .

It remains to show that $t_x t_d = t_x = t_d t_x$ for all $x \in \Gamma \cap \Gamma^{-1}$. By parts (1) and (3), $\gamma_{d,x,y} = \gamma_{x,y,d} \neq 0$ for some $y \in \Gamma \cap \Gamma^{-1}$ only if $y = x^{-1}$, and in this case $\gamma_{d,x,y} = \gamma_{d,x,x^{-1}} = \gamma_{x,x^{-1},d} = 1$. This implies $t_d t_x = t_x$. Similarly, $\gamma_{x,d,y} = \gamma_{y,x,d} \neq 0$ for some $y \in \Gamma \cap \Gamma^{-1}$ only if $y = x^{-1}$, whence $\gamma_{x,d,y} = \gamma_{x,d,x^{-1}} = \gamma_{x^{-1},x,d} = 1$ by Part (4). This implies $t_x t_d = t_x$.

- (b) Let $x, y \in E$, and suppose $\gamma_{x,y,z^{-1}} \neq 0$ for some $z \in W$. Let Γ be the left cell containing y . Then $\Gamma \subseteq E$. By part (2), $z \sim_L y$, therefore $z \in \Gamma \subseteq E$ as well, hence J_E is a subalgebra.

Now suppose $x, y \in W$ belong in different 2-sided cells, say with $x \in E$ and $y \in E'$. Then $y^{-1} \in E'$ by Proposition 2.16, hence $x \not\sim_L y^{-1}$. Part (2) now implies that $\gamma_{x,y,z^{-1}} = 0$ for all $z \in W$, therefore $t_x t_y = 0$. It follows that $J = \oplus_{E \in \mathcal{C}} J_E$.

- (c) By part (4) of Proposition 3.3, the fact that $E \cap \mathcal{D}$ is finite implies E is a disjoint union of finitely many left cells $\Gamma_1, \dots, \Gamma_k$. Suppose $\Gamma_i \cap \mathcal{D} = \{d_i\}$ for each $i \in [k]$, and let $x \in E$, say with $x \in \Gamma_i$ and $x^{-1} \in \Gamma_{i'}$ for some $i, i' \in [k]$. Then by parts (1), (2) and (3), $\gamma_{x,d_j,y} = \gamma_{y,x,d_j} \neq 0$ for some $y \in E, j \in [k]$ only if $d_j \sim_L x$ and $y = x^{-1}$. In this case, $j = i$ and $\gamma_{x,d_j,y} = \gamma_{y,x,d_i} = 1$ by part (4). Consequently,

$$t_x \left(\sum_{j=1}^k t_{d_j} \right) = t_x t_{d_i} = t_x.$$

Similarly, $\gamma_{d_j,x,y} = \gamma_{x,y,d_j}$ for some $y \in E, j \in [k]$ only if $d_j \sim_L x^{-1}$ and $y = x^{-1}$, in which case $j = i'$ and $\gamma_{d_j,x,y} = \gamma_{x,x^{-1},d_{i'}} = 1$. Consequently,

$$\left(\sum_{j=1}^k t_{d_j} \right) t_x = t_{d_{i'}} t_x = t_x.$$

It follows that $\sum_{d \in E \cap \mathcal{D}} t_d = \sum_{j=1}^k t_{d_j}$ is the unit of J_E , as claimed.

- (d) Let $x \in W$, and let d_1, d_2 be the unique distinguished involution in the left cell of x and x^{-1} , respectively. To show $\sum_{d \in \mathcal{D}} t_d$ is the unit of J , it suffices to show that

$$t_x \left(\sum_{d \in \mathcal{D}} t_d \right) = t_x t_{d_1} = t_x = t_{d_2} t_x = \left(\sum_{d \in \mathcal{D}} t_d \right) t_x.$$

This can be proved in a similar way to the last part. \square

Remark 3.7. In part (3) of the corollary, we dealt with the case where \mathcal{D} is finite. When \mathcal{D} is infinite, J only has a generalized unit element in the sense that the elements $t_d (d \in \mathcal{D})$ satisfy $t_d t_{d'} = \delta_{d,d'}$ and $\sum_{d,d' \in \mathcal{D}} t_d J t_{d'} = J$. Lusztig also showed that even when \mathcal{D} is not finite, J can be naturally imbedded into a certain unital algebra ([Lus14], 18.13). We will not need these technicalities, though.

3.3. The subregular J -ring. We are now ready to introduce our main objects of study. Consider the following sets.

Definition 3.8. Let C denote the set of all non-identity elements in W with a unique reduced expression. For each $s \in S$, let Γ_s be the sets of all elements in C whose reduced expression ends in s .

We will be interested the groups J_C and $J_{\Gamma_s \cap \Gamma_s^{-1}}$ (see Definition 3.5). Thanks to the following theorem and Corollary 3.6, they are actually subalgebras of J .

Theorem 3.9 ([Lus83], 3.8). *The set C is a 2-sided Kazhdan-Lusztig cell of W , and Γ_s is a left Kazhdan-Lusztig cell of W for each $s \in S$.*

Definition 3.10. We call the cell C the *subregular cell* of W , and we call the subalgebra J_C the *based ring of the subregular cell* of (W, S) , or simply the *subregular J -ring* of (W, S) . For each $s \in S$, we write $J_s := J_{\Gamma_s \cap \Gamma_s^{-1}}$.

The rest of the paper is devoted to the study of the algebras J_C and $J_s (s \in S)$. These algebras naturally possess the additional structures of a *based ring*. We explain this below.

The following three definitions are taken from Chapter 3 of [EGNO15].

Definition 3.11 (\mathbb{Z}_+ -rings). Let A be a ring which is free as a \mathbb{Z} -module.

- (1) A \mathbb{Z}_+ -*basis* of A is a basis $B = \{t_i\}_{i \in I}$ such that for all $i, j \in I$, $t_i t_j = \sum_{k \in I} c_{ij}^k t_k$ where $c_{ij}^k \in \mathbb{Z}_{\geq 0}$ for all $k \in I$.
- (2) A \mathbb{Z}_+ -*ring* is a ring with a fixed \mathbb{Z}_+ -basis and with identity 1 which is a nonnegative linear combination of the basis elements.
- (3) A *unital* \mathbb{Z}_+ -ring is a \mathbb{Z}_+ -ring such that 1 is a basis element.

Let A be a \mathbb{Z}_+ -ring, and let I_0 be the set of $i \in I$ such that t_i occurs in the decomposition of 1. We call the elements of I_0 the *distinguished index set*. Let $\tau : A \rightarrow \mathbb{Z}$ denote the group homomorphism defined by

$$\tau(t_i) = \begin{cases} 1 & \text{if } i \in I_0, \\ 0 & \text{if } i \notin I_0. \end{cases}$$

Definition 3.12 (Based rings). A \mathbb{Z}_+ -ring A with a basis $\{t_i\}_{i \in I}$ is called a *based ring* if there exists an involution $i \mapsto i^*$ such that the induced map

$$a = \sum_{i \in I} c_i t_i \mapsto a^* := \sum_{i \in I} c_i t_{i^*}, c_i \in \mathbb{Z}$$

is an anti-involution of the ring A , and

$$(6) \quad \tau(t_i t_j) = \begin{cases} 1 & \text{if } i = j^*, \\ 0 & \text{if } i \neq j^*. \end{cases}$$

Definition 3.13 (Multifusion rings and fusion rings). A *multifusion ring* is a based ring of finite rank. A *fusion ring* is a unital based ring of finite rank.

We now use results from §3.2 to show that under certain finiteness conditions, all the subalgebras of J introduced in the subsection are based rings.

Proposition 3.14. (1) *Let E be any 2-sided KL cell in W that contains finitely many distinguished involutions. Then the algebra J_E is a based ring with basis $\{t_x\}_{x \in I}$ with index set $I = E$, with distinguished index set $I_0 = E \cap \mathcal{D}$, and with the map $*$: $I \rightarrow I$ given by $x^* = x^{-1}$.*
 (2) *Let Γ be any left KL cell in W , and let d be the unique element in $\Gamma \cap \mathcal{D}$. Then $J_{\Gamma \cap \Gamma^{-1}}$ is a unital based ring with index set $I = \Gamma \cap \Gamma^{-1}$, with distinguished index set $I_0 = \{d\}$, and with $*$: $I \rightarrow I$ given by $x^* = x^{-1}$.*

Proof. (1) The set $\{t_x\}_{x \in E}$ forms a \mathbb{Z}_+ -basis of J_E by the definition of J_E , and J_E is \mathbb{Z}_+ -ring with distinguished index set $E \cap \mathcal{D}$ since the its unit is $\sum_{d \in E \cap \mathcal{D}} t_d$ by Part (c) of Corollary 3.6. J_E . The fact that $x \mapsto x^{-1}$ induces an anti-involution on J_E follows from Corollary 3.2. Finally, Equation (6) holds by parts (3) and (4) of Proposition 3.3. We have now shown that J_E is a based ring.

(2) The proof is similar to the previous part, with the only difference being that $J_{\Gamma \cap \Gamma^{-1}}$ is unital with $I_0 = \{d\}$ since t_d is its unit by Part (a) of Corollary 3.6. \square

Corollary 3.15. *Let (W, S) be a Coxeter system where S is finite (this will be the case for all Coxeter systems in this paper). Let C, Γ_s, J_C and J_s be as before. Then*

- (1) J_C is a based ring with index set $I = C$, distinguished index set $I_0 = S$ anti-involution induced by the map $*$: $I \rightarrow I$ with $x^* = x^{-1}$.
- (2) For each $s \in S$, J_s is a based ring with index set $I = C$, distinguished index set $I_0 = \{s\}$ anti-involution induced by the map $*$: $I \rightarrow I$ with $x^* = x^{-1}$.

Proof. This is immediate from Proposition 3.14 and Theorem 3.9 once we show that for each $s \in S$, the unique distinguished involution in Γ_s is exactly s . So it suffices to show that $s \in \mathcal{D}$ for each $s \in S$. This is well-known (we will also see this from the proof of Corollary 4.10, where we show $\mathbf{a}(s) = \Delta(s) = 1$ for all $s \in S$). \square

For future use, let us formulate the notion of an isomorphism of based rings. Naturally, we define it to be a ring isomorphism that respects all the additional defining structures of a based ring.

Definition 3.16 (Isomorphism of Based Rings). Let A be a based ring $\{t_i\}_{i \in I}$ with index set I , distinguished index set I_0 and anti-involution $*$ induced by a map $*$: $I \rightarrow I$. Let B be a based ring $\{t_j\}_{j \in J}$ with index set J , distinguished index set J_0 and anti-involution $*$ induced by a map $*$: $J \rightarrow J$. We define an *isomorphism of based rings* from A to B to be a unit-preserving ring isomorphism $\Phi : A \rightarrow B$ such that $\Phi(t_i) = t_{\phi(i)}$ for all $i \in I$, where ϕ is a bijection from I to J such that $\phi(I_0) = J_0$ and $\Phi(t_i^*) = (\Phi(t_i))^*$ for all $i \in I$.

4. PRODUCTS IN J_C

The notations from the previous sections remain in force. In particular, we assume (W, S) is an arbitrary Coxeter system with $S = [n]$ for some $n \in \mathbb{N}$, and we use C to denote the subregular cell.

In this section, we develop the tools to study the algebra J_C . The notion of the *dihedral segments* of a word plays a central role. First, we use this notion to characterize elements in C , and enumerate basis elements of J_C as walks on certain graphs. Second, we prove Theorem F, and reduce the study of a basis element t_w in J_C to only the basis elements corresponding to its dihedral segments. Finally, we explain how to use Theorem F to compute the products of arbitrary basis elements in J_C . The next two sections of the paper will depend heavily on combining our knowledge of the basis elements in J_C and our ability to compute their products.

4.1. Dihedral segments. We characterize the elements of the subregular cell C in terms of their reduced words. Since no simple reflection can appear consecutively in a reduced word of any element in W , we make the following assumption.

Assumption 4.1. *From now on, whenever we speak of a word in a Coxeter system, we assume that no simple reflection appears consecutively in the word.*

With this assumption, we may now define the dihedral segments of a word.

Definition 4.2 (Dihedral segments). For any word $x \in \langle S \rangle$, we define the *dihedral segments* of x to be the maximal contiguous subwords of x involving two letters.

For example, suppose $S = [3]$ and $x = 121313123$, then x has dihedral segments $x_1 = 121, x_2 = 13131, x_3 = 12, x_4 = 23$. We may think of breaking a word into its dihedral segments as a “factorization” process. The process can be easily reversed, that is, we may recover a word from its dihedral segments by taking a proper “product”. This motivates the following definition.

Definition 4.3 (Glued product). For any two words $x_1, x_2 \in \langle S \rangle$ such that x_1 ends with the same letter that x_2 starts with, say $x_1 = \cdots st$ and $x_2 = tu \cdots$, we define their *glued product* to be the word $x_1 * x_2 := \cdots stu \cdots$ obtained by concatenating x_1 and x_2 then deleting one occurrence of the common letter.

Note that the operation \cdot is obviously associative. Further, if x_1, x_2, \dots, x_k are the dihedral segments of x , then

$$(7) \quad x = x_1 * x_2 * \cdots * x_k.$$

For example, with x, x_1, x_2, x_3, x_4 as before, we have

$$x_1 * x_2 * x_3 * x_4 = 121 * 13131 * 12 * 23 = 121313123 = x.$$

Clearly, the dihedral segments of a word must alternate in two letters and take the form $sts \cdots$ for some $s, t \in S$. It is thus convenient to have the following notation.

Definition 4.4. For $s, t \in S$ and $k \in \mathbb{N}$, let $(s, t)_k$ denote the alternating word $sts \cdots$ of length k . In particular, take $(s, t)_0$ to be the empty word \emptyset .

Proposition 2.4 now implies the following.

Proposition 4.5 (Subregular Criterion). *Let $x \in \langle S \rangle$. Then x is the reduced word of an element in C if and only no letter in S appears consecutively in x and each dihedral segment of x is of the form $(s, t)_k$ for some $s, t \in S$ and $k < m(s, t)$.*

We can use this criterion to enumerate the elements of C .

Definition 4.6 (Subregular graph). Let $H, T : S^* \setminus \{\emptyset\} \rightarrow S$ be the functions that send any nonempty word $w = s_1 s_2 \cdots s_k$ to its first and last letter s_1 and s_k , respectively. Let $D = (V, E)$ be the directed graph such that

- (1) $V = \{(s, t)_k : s, t \in S, 0 < k < m(s, t)\}$,
- (2) E consists of directed edges (v, w) pointing from v to w , where
 - (a) either $v = (s, t)_{k-1}$ and $w = (s, t)_k$ for some $s, t \in S, 1 < k < m(s, t)$,
 - (b) or v and w are alternating words containing different sets of letters, yet $T(v) = H(w)$.

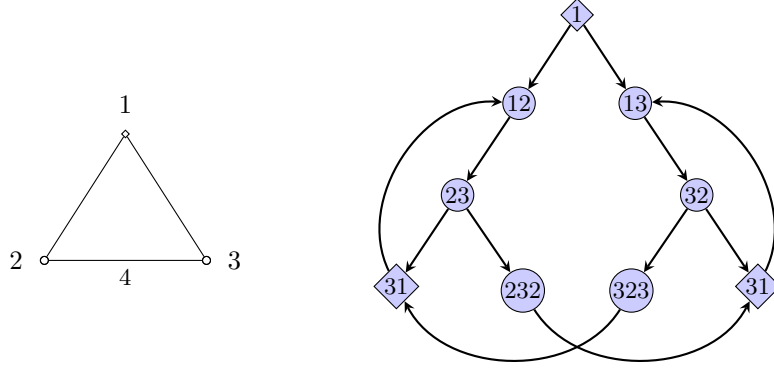
We call the graph D the *subregular graph* of (W, S) .

Recall that a *walk* on a directed graph is a sequence of vertices v_1, v_2, \dots, v_k such that (v_i, v_{i+1}) is an edge for all $1 \leq i \leq k-1$. Note that walks on D correspond bijectively to elements of C . Indeed, given any walk v_1, v_2, \dots, v_k on D , imagine we successively write down the words $x_i = T(v_1) \cdots T(v_i)$ as we traverse the walk. Then the vertex v_i records exactly the dihedral segment at the end of x_i , and traveling along an edge of type (b) to v_{i+1} corresponds to starting a new dihedral segment, while traversing an edge of type (a) corresponds to extending the last dihedral segment of the word x_i by one more letter.

Under the bijection described above, it is also easy to see that for any $s \in S$, the elements of $\Gamma_s \cap \Gamma_s^{-1}$ correspond to walks on the subregular graph D that starts at the vertex $v = (s)$ (an alternating word of length 1) and ends at a vertex w with $T(w) = s$. Call such a walk an *s-walk*. Such walks often involve only a subset V' of the vertex set V of D , in which case they are exactly walks on the subgraph of D induced by V' . We denote this subgraph by D_s .

Remark 4.7. When $m(s, t) < \infty$ for all $s, t \in S$, the vertex sets of D and D_s ($s \in S$) are necessarily finite, hence D and D_s can be viewed as *finite state automata* that recognize C and $\Gamma_s \cap \Gamma_s^{-1}$, respectively, in the sense of formal languages (see [AH74]).

Example 4.8. Let (W, S) be the Coxeter system whose Coxeter diagram is the triangle in Figure 4.8. It is easy to see that D_1 should be the directed graph on the right, and elements of $\Gamma_1 \cap \Gamma_1^{-1}$ correspond to walks on D_1 that start with the top vertex and end with either the bottom-left or bottom-right vertex.



4.2. \mathbf{a} -function characterization of C . We give yet another characterization of the subregular cell C , this time in terms of the \mathbf{a} -function defined in §3.1. To start, we recall some properties of \mathbf{a} .

Proposition 4.9 ([Lus14], 13.7, 14.2). *Let $x, y \in W$. Then*

- (1) $\mathbf{a}(x) \geq 0$, where $\mathbf{a}(x) = 0$ if and only if x equals the identity element of W .
- (2) $\mathbf{a}(x) \leq \Delta(x)$.
- (3) If $x \leq_{LR} y$, then $\mathbf{a}(x) \geq \mathbf{a}(y)$. Hence, if $x \sim_{LR} y$, then $\mathbf{a}(x) = \mathbf{a}(y)$.
- (4) If $x \leq_L y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_L y$.
- (5) If $x \leq_R y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_R y$.
- (6) If $x \leq_{LR} y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_{LR} y$.

Corollary 4.10. $C = \{x \in W : \mathbf{a}(x) = 1\}$.

Proof. Let $s \in S$. Then $\mathbf{a}(s) \geq 1$ by Part (1) of the proposition. On the other hand, it is well known that $c_s = T_s + v^{-1}$ ([Lus14], § 5), therefore $\Delta(s) = 1$ by the definition of Δ and $\mathbf{a}(s) \leq 1$ by part (2) of the proposition. It follows that $\mathbf{a}(s) = 1$. Since s is clearly in C , Part (3) implies that $\mathbf{a}(x) = 1$ for all $x \in C$.

Now let $x \in W \setminus C$. Then either x is the group identity and $\mathbf{a}(x) = 0$, or x has a reduced expression $x = s_1 s_2 \cdots s_k$ with $k > 1$ and each $s_i \in S$. In the latter case, $x \leq_L s_k$ by Proposition 2.13, so $\mathbf{a}(x) \geq \mathbf{a}(s_k) = 1$. Meanwhile, since $x \notin C$, $x \not\sim_{LR} s_k$, so $\mathbf{a}(x) \neq \mathbf{a}(s_k)$ by part (6) of Proposition 4.9. It follows that $\mathbf{a}(x) > 1$, and we are done. \square

The characterization leads to a shortcut for studying products in J_C . To see how, consider the filtration

$$\cdots \subset H_{\geq 2} \subset H_{\geq 1} \subset H_{\geq 0} = H.$$

of the Hecke algebra H where

$$H_{\geq a} = \bigoplus_{x: \mathbf{a}(x) \geq a} \mathcal{A}c_x$$

for each $a \in \mathbb{N}$. By parts (3)-(6) of Proposition 4.9 and Proposition 2.14, this may be viewed as a filtration of submodules when we view H as its regular left module. It induces the left modules

$$(8) \quad H_a := H_{\geq a} / H_{\geq a+1},$$

where H_a is spanned by images of the elements $\{c_x : \mathbf{a}(x) = a\}$. In particular, H_1 is spanned by the images of $\{c_x : x \in C\}$. By the construction of J , to compute a product $t_x \cdot t_y$ in J_C , it then suffices to consider the product $c_x \cdot c_y$ in H_1 . More precisely, we have arrived at the following shortcut.

Corollary 4.11. *Let $x, y \in C$. Suppose*

$$c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$$

for $h_{x,y,z} \in \mathcal{A}$. Then

$$t_x t_y = \sum_{z \in T} \gamma_{x,y,z^{-1}} t_z$$

in J_C , where $T = \{z \in C : h_{x,y,z} \in n_z v + \mathbb{Z}[v^{-1}] \text{ for some } n_z \neq 0\}$.

The corollary plays a key role in the proof of Lemma 4.15. A simple application of it reveals the following, which we will use repeatedly in the next section.

Corollary 4.12. *Let $x = s_1 s_2 \cdots s_k$ be the reduced word of an element in C . Then*

$$t_{s_1} t_x = t_x = t_x t_{s_k}.$$

Proof. This follows immediately from Corollary 4.11 and Proposition 2.13. \square

4.3. The dihedral factorization theorem. Recall the definition of dihedral segments from §4.1. This subsection is dedicated to the proof of Theorem 1. We restate it below.

Theorem F. (Dihedral factorization) *Let x be the reduced word of an element in C , and let x_1, x_2, \dots, x_l be the dihedral segments of w . Then*

$$t_x = t_{x_1} \cdot t_{x_2} \cdots t_{x_l}.$$

It is convenient to have the following definition.

Definition 4.13. (Dihedral elements) We define a *dihedral element* in J_C to be a basis element of the form t_x , where x appears as a dihedral segment of some $y \in C$.

In light of the definition, the theorem means that dihedral elements generate J_C . The theorem also means that the combinatorial factorization of x into its dihedral segments in Equation 7 carries over to an algebraic one in J_C .

To prove the theorem, we need to examine products in H and apply Corollary 4.11. To fully exploit the uniqueness of reduced expressions of elements of C , we need the following well-known fact.

Proposition 4.14 ([KL79], Statement 2.3.e). *Let $x, y \in W, s \in S$ be such that $x < y, sy < y, sx > x$. Then $\mu(x, y) \neq 0$ if and only if $x = sy$; further, in this case, $\mu(x, y) = 1$.*

Lemma 4.15. *Let $x = s_1 s_2 s_3 \cdots s_k$ be the reduced word of an element in C . Let $x' = s_2 s_3 \cdots s_k$ and $x'' = s_3 \cdots s_k$ be the sequences obtained by removing the first letter and first two letters from x , respectively. Then in H_1 , we have*

$$c_{s_1} c_{x'} = \begin{cases} c_{x''} & \text{if } s_1 \neq s_3; \\ c_x + c_{x''} & \text{if } s_1 = s_3. \end{cases}$$

Proof. By Proposition 2.13 and Corollary 4.10, in H_1 we have

$$c_s c_{x'} = c_x + \sum_P \mu_{z,x'} c_z$$

where $P = \{z \in C : s_1 z < z < x'\}$. Let $z \in P$. Then z has a unique reduced expression that is a proper subword of x' and starts with s_1 . Since $s_1 \neq s_2$ now that x is reduced, we have $\mathcal{L}(z) = \{s_1\}$, therefore $s_2 x' < x'$ while $s_2 z > z$. Now, if $l(z) < l(x') - 1$, then $z \neq s_2 x$, so $\mu(z, x') = 0$ by Lemma 4.14. If $l(z) = l(x') - 1$, then we must have $s_3 = s_1$ and $z = x'' = s_2 x'$, for otherwise $s_2 \neq s_1, s_3 \neq s_1$, and any subword of $x' = s_2 s_3 \cdots s_k$ that starts with s_1 must have length smaller than $l(x') - 1$. This implies $\mu(z, x') = 1$ by Lemma 4.14. The lemma now follows. \square

We are ready to prove the theorem.

Proof of Theorem F. We use induction on l . The base case where $l = 1$ is trivially true. If $l > 1$, let y be the glued product $y = x_2 * x_3 * \cdots * x_l$, so that by induction, it suffices to show

$$(9) \quad t_x = t_{x_1} \cdot t_y.$$

Suppose y starts with some $t \in S$. Note that the construction of the dihedral segments guarantees that x_1 contains at least two letters and is of the alternating form $w_1 = \cdots t s t$ for some $s \in S$, while x_2 , hence also y , is of the form $t u \cdots$ for some $u \in S \setminus \{s, t\}$.

We prove Equation (9) by induction on the length $k = l(x_1)$ of x_1 . For the base case $k = 2$, Proposition 2.13 and Lemma 4.15 imply that

$$c_{x_1} c_y = c_{s t} c_{t u \cdots} = c_s c_t c_{t u \cdots} = (v + v^{-1}) c_{s t u \cdots} = (v + v^{-1}) c_{x_1 * y}$$

in H_1 . Equation (9) then follows by Corollary 4.11. Now suppose $k > 2$, write $x_1 = s_1 s_2 s_3 \cdots s_k$, and let $x'_1 = s_2 s_3 \cdots s_k$ and $x''_1 = s_3 \cdots s_k$. Since the letters s_1, s_2, \dots, s_k alternate between s_1 and s_2 , Proposition 2.13 and Lemma 4.15 imply that

$$c_{s_1 s_2} \cdot c_{x'_1} = c_{s_1} c_{s_2} c_{x'_1} = (v + v^{-1}) c_{s_1} c_{x'_1} = (v + v^{-1}) (c_{x_1} + c_{x''_1})$$

and similarly

$$c_{s_1 s_2} \cdot c_{x'_1 * y} = (v + v^{-1}) (c_{x_1 * y} + c_{x''_1 * y}).$$

From the last two equations, it follows that

$$\begin{aligned} t_{s_1 s_2} t_{x'_1} &= t_{x_1} + t_{x''_1}, \\ t_{s_1 s_2} t_{x'_1 \cdot y} &= t_{x_1 \cdot y} + t_{x''_1 \cdot y}, \end{aligned}$$

therefore

$$t_{x_1} t_y = (t_{s_1 s_2} t_{x'_1} - t_{x''_1}) t_y = t_{s_1 s_2} t_{x'_1 \cdot y} - t_{x''_1 * y} = t_{x_1 * y} + t_{x''_1 * y} - t_{x''_1 * y} = t_{x_1 * y} = t_x,$$

where the second equality holds by the inductive hypothesis now that $l(x'_1) < l(x_1)$. This completes our proof. \square

4.4. Products of dihedral elements. Now that Theorem F allows us to factor any basis element in J_C into dihedral elements, to understand products of basis elements, it is natural to first study products of dihedral elements. We do so now. Fortunately, for dihedral elements $t_x, t_y \in J_C$, the product $c_x c_y$ of the corresponding KL basis elements are well understood in the Hecke algebra, so the formula for $t_x t_y$ will be easy to derive. Also, as we shall see in the next subsection, we only need to focus on the case where x and y are generated by the same set of two simple reflections and x ends with the same letter that y starts with.

We need more notation. Fix $s, t \in S$. For any $k \in \mathbb{N}$, set $s_k = s t s \cdots$ to be the word that has length k , alternates in s, t and starts with s . Similarly, define $k s = (s_k)^{-1}$ to be the word of length k that alternates in s, t and ends with s .

If s_k ends with a letter $u \in \{s, t\}$ and we wish to emphasize this fact, write $s_k u$ for s_k . Similarly, if ${}_k s$ starts with $u \in \{s, t\}$, we may write $u {}_k s$ for ${}_k s$. Define the counterparts of all these words with s replaced by t in the obvious way. The following fact is well-known.

Proposition 4.16. *Let $M = m(s, t)$. Suppose $x = u {}_k s$ and $y = s {}_l u'$ for some $u, u' \in \{s, t\}$ and $0 < k, l < M$. For $d \in \mathbb{Z}$, let $\phi(d) = k + l - 1 - 2d$. Then*

$$c_x c_y = c_{u {}_k s} c_{s {}_l u'} = (v + v^{-1}) \sum_{d=\max(k+l-M, 0)}^{\min(k, l)-1} c_{u_{\phi(d)} u'} + \varepsilon$$

in H , where $\varepsilon = f \cdot c_{1_M}$ for some $f \in \mathcal{A}$ if $M < \infty$ and $\varepsilon = 0$ otherwise.

By Corollary 4.11, this immediately yields the multiplication formula below.

Proposition 4.17. *Suppose $x = u {}_k s$ and $y = s {}_l u'$ for some $u, u' \in \{s, t\}$ and $0 < k, l < M$. For $d \in \mathbb{Z}$, let $\phi(d) = k + l - 1 - 2d$. Then in J_C , we have*

$$t_x t_y = t_{u {}_k s} t_{s {}_l u'} = \sum_{d=\max(k+l-M, 0)}^{\min(k, l)-1} t_{u_{\phi(d)} u'}.$$

The obvious counterparts of the propositions with s replaced with t hold as well.

Let us decipher the formula from Proposition 4.17. It says that the product $t_x t_y$ is the linear combination of the terms t_z , all with coefficient 1, where z runs through the elements in C whose reduced words begins with the same letter as x , ends with the same letter as y , and have lengths from the list obtained in the following way: consider the list of numbers $|k - l| + 1, |k - l| + 3, \dots, k + l - 1$ of the same parity, then delete from it all numbers r with $r \geq M$, as well as their mirror images with respect to the point M , i.e., delete $2M - r$. Note that when $k = 1$, this agrees with Corollary 4.12.

Example 4.18 (Product of dihedral elements). Let $s, t \in S$.

(1) Suppose $m(s, t) = 7, x = stst$ and $y = tst$. Then by Proposition 4.17,

$$t_x t_y = t_{st} + t_{stst} + t_{ststst}.$$

(2) Suppose $m(s, t) = 7, x = stst$ and $y = tsts$. Then by Proposition 4.17,

$$t_x t_y = t_s + t_{sts} + t_{ststs} + \cancel{t_{stststs}} = t_s + t_{sts} + t_{ststs}.$$

(3) Suppose $m(s, t) = 7, x = tst$ and $y = tsts$. Then by Proposition 4.17,

$$t_x t_y = t_{tsts} + \cancel{t_{tststs}} + \cancel{t_{tstststs}} = t_{tsts}.$$

The rule we described before the example to get the list of lengths for the z 's is well-known; it is the *truncated Clebsch-Gordan rule*. It governs the multiplication of the basis elements of the *Verlinde algebra of the Lie group $SU(2)$* , which appears as the Grothendieck ring of certain fusion categories (see [EK95] and Section 4.10 of [EGNO15]). Since it will cause no confusion, we will also refer to this algebra simply as the *Verlinde algebra*.

Definition 4.19 (The Verlinde algebra, [EK95]). Let $M \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. The M -th *Verlinde algebra* is the free abelian group $\text{Ver}_M = \oplus_{1 \leq k \leq M-1} \mathbb{Z} L_k$, with multiplication defined by

$$L_k L_l = \sum_{d=\max(k+l-M, 0)}^{\min(k, l)-1} L_{k+l-1-2d}.$$

We call the \mathbb{Z} -span of the elements L_k where k is an odd integer the *odd part* of Ver_M , and denote it by $\text{Ver}_M^{\text{odd}}$.

Note that by the multiplication formula, $\text{Ver}_M^{\text{odd}}$ is clearly a subalgebra of Ver_M . Indeed, suppose (W, S) is a dihedral system, say with $S = \{1, 2\}$ and $m(1, 2) = M$ for some $M \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$, then we claim that the subalgebra J_1 of J_C is isomorphic to $\text{Ver}_M^{\text{odd}}$. To see this, recall that J_1 is given by the \mathbb{Z} -span of all t_{1_k} where k is odd, $0 < k < M$, and 1_k is the alternating word $121 \cdots 1$ containing k letters. Since the multiplication of such basis elements are governed by the truncated Clebsch-Gordan rule in Proposition 4.17, the map $t_{1_k} \mapsto L_k$ induces an isomorphism. Furthermore, it is easy to check that both Ver_M and $\text{Ver}_M^{\text{odd}}$ are unital based rings with L_1 as the unit and with the identity map as the anti-involution, so this isomorphism is actually an isomorphism of based rings. By a similar argument, J_2 is isomorphic to $\text{Ver}_M^{\text{odd}}$ as based rings as well. We discuss incarnations of $\text{Ver}_M^{\text{odd}}$ for some small values of M below.

Example 4.20. Let (W, S) be a dihedral system with $S = \{1, 2\}$ and $M = m(1, 2)$.

- (1) Suppose $M = 5$. Then $J_1 = \mathbb{Z}t_1 \oplus \mathbb{Z}t_{121}$, where t_1 is the unit and

$$t_{121}t_{121} = t_1,$$

so J_1 , hence $\text{Ver}_5^{\text{odd}}$, is isomorphic to the *Ising fusion ring* that arises from the Ising model of statistical mechanics.

- (2) Suppose $M = 6$. Then $J_1 = \mathbb{Z}t_1 \oplus \mathbb{Z}t_{121} \oplus \mathbb{Z}t_{12121}$, where t_1 is the unit and

$$t_{121}t_{121} = t_1 + t_{121} + t_{12121}, \quad t_{121}t_{12121} = t_{12121}t_{121} = t_{121}, \quad t_{12121}t_{12121} = t_1.$$

On the other hand, the category \mathcal{C} of complex representations of the symmetric group S_3 has three non-isomorphic simple objects 1 (the trivial representation), χ (the sign representation) and V satisfying

$$1 \otimes \chi = \chi \otimes 1 = \chi, \quad 1 \otimes V = V \otimes 1 = V,$$

$$V \otimes V = 1 \oplus V \oplus \chi, \quad V \otimes \chi = \chi \otimes V = V, \quad \chi \otimes \chi = 1,$$

so J_1 , hence $\text{Ver}_6^{\text{odd}}$, is isomorphic to the Grothendieck ring $\text{Gr}(\mathcal{C})$ of \mathcal{C} .

4.5. Products of arbitrary elements. Let $x, y \in C$. We now describe the product $t_x t_y$ of two arbitrary basis elements in J_C . For convenience, let us make the following assumption.

Assumption 4.21. *From now on, whenever we write $x \in C$, we assume not only that x is an element of the subregular cell, but also that x is the unique reduced word of the element.*

Recall the definition of J_X for $X \subseteq W$ from Definition 3.5 and Γ_s ($s \in S$) from the beginning of §3.3. Here is a simple fact about $t_x t_y$:

Proposition 4.22. *Let $a, b, c, d \in S$, let $x \in \Gamma_a^{-1} \cap \Gamma_b$, and let $y \in \Gamma_c^{-1} \cap \Gamma_d$. Then $t_x t_y = 0$ if $b \neq c$, and $t_x t_y \in J_{\Gamma_a^{-1} \cap \Gamma_d}$ if $b = c$.*

Proof. Recall that for any $s \in S$, Γ_s is a left KL cell in W that consists of the elements in C whose reduced word ends in s . Consequently, Γ_s^{-1} is a right KL cell by Proposition 2.15 and it consists of the elements in C whose reduced word starts with s . That said, the statement follows from part (2) of Proposition 3.3 in the following way. If $b \neq c$, then $x \in \Gamma_b$ while $y^{-1} \in \Gamma_c$, so $x \not\sim_L y^{-1}$. This implies $\gamma_{x,y,z^{-1}} = 0$ for all $z \in W$, therefore $t_x t_y = 0$. If $b = c$, then for any $z \in W$ such that $\gamma_{x,y,z^{-1}} \neq 0$, we must have $y \sim_L z$ and $z^{-1} \sim_L x^{-1}$. The last condition implies $z \sim_R x$ by Proposition 2.15, so $z \in \Gamma_a^{-1} \cap \Gamma_d$. It follows that $t_x t_y \in J_{\Gamma_a^{-1} \cap \Gamma_d}$. \square

Remark 4.23. The proposition may be interpreted as saying that for any basis element t_z that occurs in the product $t_x t_y$ (when the product is nonzero), z must start with the same letter as x and end with the same letter as y . This fact will be used later in §5.2.

For a more detailed description of $t_x t_y$, we discuss three cases. The first case simply paraphrases the case $b \neq c$ in Proposition 4.22.

Proposition 4.24. *Let $x, y \in C$. Suppose x does not end with the letter that y starts with. Then $t_x t_y = 0$.*

Proof. This is immediate from Proposition 4.22. \square

The second case is also relatively simple.

Proposition 4.25. *Let $x, y \in C$. Suppose x ends with the letter that y starts with, and suppose that the last dihedral segment of x and the first dihedral segment of y involve different sets of letters. Then $t_x t_y = t_{x*y}$.*

Proof. Let x_1, \dots, x_p and y_1, \dots, y_q be the dihedral segments of x and y , respectively. By the assumptions, $x_1, \dots, x_k, y_1, \dots, y_l$ are exactly the dihedral segments of the glued product $x * y$, therefore Theorem F implies

$$t_x t_y = t_{x_1} \cdots t_{x_p} t_{y_1} \cdots t_{y_q} = t_{x_1 * \cdots * x_p * y_1 * \cdots * y_q} = t_{x*y}. \quad \square$$

For the third and most involved case, it remains to compute products of the form $t_x t_y$ in J_C where x ends in the letter y starts with and the last dihedral segment x_p of x contain the same set of letters as the first dihedral segment y_1 of y . By Theorem F, to understand $t_x t_y$ we need to first understand $t_{x_p} t_{y_1}$. This leads us to the configuration studied in Proposition 4.17. We illustrate below how we may combine Proposition 4.17 and Theorem F to compute $t_x t_y$.

Example 4.26 (Product of arbitrary elements). Suppose $S = \{1, 2, 3\}$ and $m(1, 2) = 4, m(1, 3) = 5, m(2, 3) = 6$.

(1) Let $x = 123, y = 323213$. Then by Theorem F and Proposition 4.17,

$$\begin{aligned} t_x t_y &= t_{12} t_{23} t_{3232} t_{21} t_{13} \\ &= t_{12} (t_{232} + t_{23232}) t_{21} t_{13} \\ &= t_{12} t_{232} t_{21} t_{13} + t_{12} t_{23232} t_{21} t_{13}. \end{aligned}$$

Applying Theorem 1 again to the last expression, we have

$$t_x t_y = t_{123213} + t_{12323213}.$$

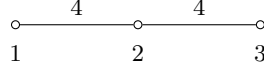
(2) Let $x = 123, y = 3213$. Repeated use of Theorem F and Proposition 4.17 yields

$$\begin{aligned} t_x t_y &= t_{12} t_{23} t_{32} t_{21} t_{13} \\ &= t_{12} (t_2 + t_{232}) t_{21} t_{13} \\ &= (t_{12} t_2) t_{21} t_{13} + t_{12} t_{232} t_{21} t_{13} \\ &= (t_{12} t_{21}) t_{13} + t_{12} t_{232} t_{21} t_{13} \\ &= (t_1 + t_{121}) t_{13} + t_{12} t_{232} t_{21} t_{13} \\ &= t_1 t_{13} + t_{121} t_{13} + t_{12} t_{232} t_{21} t_{13} \\ &= t_{13} + t_{1213} + t_{123213}. \end{aligned}$$

The examples illustrate the general algorithm to compute the product $t_x t_y$ in our third case. Namely, we first compute $t_{x_p} t_{y_1}$ and distribute the product so as to write $t_x t_y$ as a linear combinations of products of dihedral elements. If such a product has two consecutive factors corresponding to elements in the same dihedral group, use Proposition 4.17 to compute the product of these two factors first, then distribute the product to obtain a new linear combination. Repeat this process until we have a linear combination of products where no consecutive factors correspond to elements of the same dihedral group. This means the factors appear as the dihedral segment of an element in C , so we may apply Theorem F to each of the

products and rewrite $t_x t_y$ as a linear combination of other basis elements. Some Sage ([Dev16]) code implementing this algorithm is available at [Xu].

Example 4.27. Consider the algebra J_1 arising from the Coxeter system with the following diagram.



Let $x = 121$, $y = 12321$, and let y_n denote the glued product $y * y * \dots * y$ of n copies of y for each $n \in \mathbb{Z}_{\geq 1}$. It is easy to see that $\Gamma_1 \cap \Gamma_1^{-1}$ consists exactly of 1 , x and all y_n where $n \geq 1$ so that J_1 has basis elements t_1, t_x and t_n ($n \geq 1$) where we set $t_n := t_{y_n}$ for all $n \geq 1$. One efficient way to see this is to draw the subgraph D_1 of the subregular graph of the system (see Definition 4.6 and the ensuing discussion) shown in Figure 4.5 and recall that elements of $\Gamma_1 \cap \Gamma_1^{-1}$ are in a bijection with the walks on D_1 which start at the top vertex and end at one of the diamond-shaped vertices.

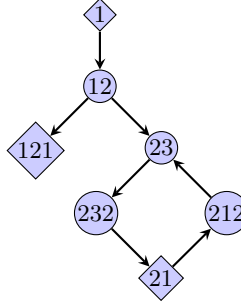


FIGURE 1. The graph D_1

Let us describe the products of all pairs of basis element in J_1 . First, we have $t_1 t_w = t_w = t_w t_1$ for each basis element $t_w \in J_1$, as t_1 is the identity. For products involving t_x but not t_1 , propositions 4.25 and 4.17 imply that $t_x t_x = t_{121} t_{121} = t_1$, while

$$(10) \quad t_x t_n = t_{121} t_{12321} \dots = t_{121} t_{12} t_{2321} * y_{n-1} = t_{12} t_{2321} * y_{n-1} = t_n$$

and similarly $t_n t_x = t_n$ for all $n \geq 1$ (where we set $y_0 = 1$). Finally, to describe products of the form $t_m t_n$ where $m, n \geq 1$, set $t_0 = t_1 + t_x$. Using computations similar to those in Equation (10), we can easily check that $y_1 y_n = y_{n-1} + y_{n+1}$ for all $n \geq 1$, then show by induction on m that

$$(11) \quad t_m t_n = t_{|m-n|} + t_{m+n}$$

for all $m, n \geq 1$.

5. J_C AND THE COXETER DIAGRAM

Let (W, S) be an arbitrary Coxeter system, and let J_C be its subregular J -ring. We study the relationship between J_C and the Coxeter diagram of (W, S) in this section.

5.1. Simply-laced Coxeter systems. Let us recall some graph-theoretic terminology. Let $G = (V, E)$ be an undirected graph. Recall that just as in the directed case, a *walk* on G is a sequence $P = (v_1, \dots, v_k)$ of vertices in G such that $\{v_i, v_{i+1}\}$ is an edge for all $1 \leq i \leq k-1$. We define a *spur* on G to be a walk of the form (v, v', v) where $\{v, v'\}$ forms an edge. Given any walk containing a spur, i.e., a walk of the form $P_1 = (\dots, u, v, v', v, u', \dots)$, we may remove the spur to form a new walk $P_2 = (\dots, u, v, u', \dots)$; conversely, we can add a spur (v, v', v) to a walk of the form P_2 to obtain the walk P_1 .

Recall that a *groupoid* may be viewed as a generalization of a group, in that it is defined to be a pair (\mathcal{G}, \circ) , where \mathcal{G} is set and \circ is a partially-defined binary operation on \mathcal{G} that satisfy certain axioms (see [CdSW99]). More precisely, for any topological space X and a chosen subset A of X , the *fundamental groupoid of X based on A* is defined to be $\Pi(X, A) := (\mathcal{P}, \circ)$, where \mathcal{P} are the homotopy equivalence classes of paths on X that connect points in A and \circ is concatenation of paths. Given an undirected graph $G = (V, E)$, we may view G as embedded in a topological surface and hence as a topological space with the subspace topology induced from the surface. We define the *fundamental groupoid of G* to be $\Pi(G) := \Pi(G, V) = (\mathcal{P}, \circ)$, where \mathcal{P} stands for paths on G .

Note that paths on G are just walks, and concatenation of paths correspond to concatenation of walks. More precisely, for any two walks $P = (v_1, \dots, v_{k-1}, v_k)$ and $Q = (u_1, u_2, \dots, u_l)$ on G , we define their *concatenation* to be the walk $P \circ Q = (v_1, \dots, v_{k-1}, v_k, u_2, \dots, u_l)$ if $v_k = u_1$; otherwise we leave $P \circ Q$ undefined. Also note that two walks are homotopy equivalent if and only if they can be obtained from each other by a sequence of removals or additions of spurs, and each homotopy equivalence class of walks contains a unique walk with no spurs. We use $[P]$ to denote the class of a walk P . For each path $P = (v_1, v_2, \dots, v_k)$, we also define its *inverse* to be the walk $P^{-1} := (v_k, \dots, v_2, v_1)$.

For each vertex s in G , we may similarly define the *fundamental group of G based at s* to be $\Pi_s(G) = (\mathcal{P}_s, \circ)$, where \mathcal{P}_s are now equivalence classes of walks on G that start and end with s , and \circ is concatenation as before. Note that $\Pi_s(G)$ is actually a group, so it makes sense to talk about its group algebra $\mathbb{Z}\Pi_s(G)$ over \mathbb{Z} . We may define a counterpart of $\mathbb{Z}\Pi_s(G)$ for $\Pi(G)$ by mimicking the construction of a group algebra.

Definition 5.1. Let $\Pi(G) = (\mathcal{P}, \circ)$ be the fundamental groupoid of a graph G . We define the *groupoid algebra* of $\Pi(G)$ over \mathbb{Z} to be the free abelian group $\mathbb{Z}\mathcal{P} = \bigoplus_{[P] \in \mathcal{P}} \mathbb{Z}[P]$ equipped with an \mathbb{Z} -bilinear multiplication \cdot defined by

$$[P] \cdot [Q] = \begin{cases} [P \circ Q] & \text{if } P \circ Q \text{ is defined in } G, \\ 0 & \text{if } P \circ Q \text{ is not defined.} \end{cases}$$

Note that $\mathbb{Z}\Pi(G)$ is clearly associative.

Proposition 5.2. Let $G = (V, E)$ where V is finite. Let P_s be the constant walk (s) for all $s \in V$. Then the groupoid algebra $\mathbb{Z}\Pi(G)$ has the structure of a based ring with basis $\{[P]\}_{[P] \in \mathcal{P}}$, with unit $1 = \sum_{s \in V} [P_s]$ (so the distinguished index set simply corresponds to V), and with its anti-involution induced by the map $[P] \mapsto [P^{-1}]$. For each $s \in V$, the group algebra $\mathbb{Z}\Pi_s(G)$ has the structure of a unital based ring with basis $\{[P]\}_{[P] \in \mathcal{P}_s}$, with unit $1 = [P_s]$ (so the distinguished index set is simply $\{s\}$), and with its anti-involution induced by the map $[P] \mapsto [P^{-1}]$.

Proof. All the claims are easy to check using definitions. \square

Now, suppose (W, S) is a simply-laced Coxeter system, and let G be its Coxeter diagram. Recall that this means $m(s, t) = 3$ for $s, t \in S$ whenever $\{s, t\}$ is an edge

in G while $m(s, t) = 2$ otherwise. Let us consider the map $C \rightarrow \Pi(G)$ which sends each element $x = s_1 \cdots s_k \in C$ to the homotopy equivalence class $[P_x]$ of the walk $P_x : (s_1, s_2, \dots, s_k)$. We claim this is a bijection.

To see this, note that for each $1 \leq i \leq k-1$, since $s_i s_{i+1}$ appears inside a dihedral segment of x , Proposition 4.5 implies that $m(s_i, s_{i+1}) > 2$, therefore $\{s_i, s_{i+1}\}$ is an edge and $P_x = (s_1, \dots, s_k)$ is a walk on G . Further, we must have $m(s_i, s_{i+1}) = 3$, so $s_{i+2} \neq s_i$ for all $1 \leq i \leq k-2$ by Proposition 4.5, therefore P_x contains no spurs. This means P_x is exactly the unique representative with no spurs in its class. Conversely, given class of walks in $\Pi(G)$, we may take its unique representative (s_1, \dots, s_k) with no spurs and consider the word $s_1 \cdots s_k$. By Proposition 4.5, $s_1 \cdots s_k$ is the reduced word of an element in C . This gives a two-sided inverse to the map $x \mapsto [P_x]$.

Since C and \mathcal{P} index the basis elements of J_C and $\mathbb{Z}\Pi(G)$, respectively, the bijection $x \mapsto [P_x]$ induces a unique \mathbb{Z} -module isomorphism $\Phi : J_C \rightarrow \mathbb{Z}\Pi(G)$ defined by

$$(12) \quad \Phi(t_x) = [P_x], \quad \forall x \in C.$$

We are now ready to prove Theorem A, which is restated below.

Theorem A. *Let (W, S) be an any simply-laced Coxeter system, and let G be its Coxeter diagram. Let $\Pi(G)$ be the fundamental groupoid of G , let $\Pi_s(G)$ be the fundamental group of G based at s for any $s \in S$, let $\mathbb{Z}\Pi(G)$ be the groupoid algebra of $\Pi(G)$, and let $\mathbb{Z}\Pi_s(G)$ be the group algebra of $\Pi_s(G)$. Then $J_C \cong \mathbb{Z}\Pi(G)$ as based rings, and $J_s \cong \mathbb{Z}\Pi_s(G)$ as based rings for all $s \in S$.*

Proof. We show that the \mathbb{Z} -module isomorphism $\Phi : J_C \rightarrow \mathbb{Z}\Pi(G)$ defined by Equation 12 is an algebra isomorphism. This would imply $J_s \cong \mathbb{Z}\Pi_s(G)$ for all $s \in S$, since Φ clearly restricts to a \mathbb{Z} -module map from J_s to $\mathbb{Z}\Pi_s(G)$. The fact that Φ and the restrictions are actually isomorphisms of based rings will then be clear once we compare the based ring structure of $J_C, \mathbb{Z}\Pi(G), J_s$ and $\mathbb{Z}\Pi_s(G)$ described in Corollary 3.15 and Proposition 5.2.

To show Φ is an algebra homomorphism, we need to show

$$(13) \quad [P_x] \cdot [P_y] = \Phi(t_x t_y)$$

for all $x, y \in C$. Let $s_k \cdots s_1$ and $u_1 \cdots u_l$ be the reduced word of x and y , respectively. If $s_1 \neq u_1$, then Equation (13) holds since both sides are zero by Definition 5.1 and Corollary 4.22. If $s_1 = u_1$, let $q \leq \min(k, l)$ be the largest integer such that $s_i = u_i$ for all $1 \leq i \leq q$. Then

$$\begin{aligned} [P_x] \cdot [P_y] &= [(s_k, \dots, s_{q+1}, s_q, \dots, s_1) \circ (s_1, \dots, s_q, u_{q+1}, \dots, u_l)] \\ &= [(s_k, \dots, s_{q+1}, s_q, \dots, s_2, s_1, s_2, \dots, s_q, u_{q+1}, \dots, u_l)] \\ &= [(s_k, \dots, s_{q+1}, s_q, u_{q+1}, \dots, u_l)], \end{aligned}$$

where the last equality holds by successive removal of spurs of the form (s_{i+1}, s_i, s_{i+1}) . On the other hand, since $m(s_i, s_{i+1}) = 3$ for each $1 \leq i \leq q$, Proposition 4.17 implies

$$(14) \quad t_{s_{i+1}s_i} t_{s_i s_{i+1}} = t_{s_{i+1}}, \quad t_{s_i} t_{s_i s_{i+1}} = t_{s_i s_{i+1}},$$

therefore by Theorem F,

$$\begin{aligned} t_x t_y &= (t_{s_k \cdots s_{q+1} s_q} t_{s_q s_{q-1}} \cdots t_{s_3 s_2} t_{s_2 s_1}) (t_{s_1 s_2} t_{s_2 s_3} \cdots t_{s_{q-1} s_q} t_{s_q u_{q+1} \cdots u_l}) \\ &= (t_{s_k \cdots s_{q+1} s_q} t_{s_q s_{q-1}} \cdots t_{s_3 s_2}) t_{s_2} (t_{s_2 s_3} \cdots t_{s_{q-1} s_q} t_{s_q u_{q+1} \cdots u_l}) \\ &= (t_{s_k \cdots s_{q+1} s_q} t_{s_q s_{q-1}} \cdots t_{s_3 s_2}) (t_{s_2 s_3} \cdots t_{s_{q-1} s_q} t_{s_q u_{q+1} \cdots u_l}) \\ &= \dots \\ &= t_{s_k \cdots s_{q+1} s_q} t_{s_q u_{q+1} \cdots u_l} \\ &= t_{s_k \cdots s_{q+1} s_q u_{q+1} \cdots u_l}. \end{aligned}$$

Here the last equality follows from Proposition 4.25, and the “ \dots ” signify repeated use of the equations in (14) to “remove” the products of the form $(t_{s_{i+1}} t_{s_i})(t_{s_i} t_{s_{i+1}})$ where $2 \leq i \leq q-1$. By the definition of Φ , we then have

$$\Phi(t_x t_y) = [(s_k, \dots, s_{q+1}, s_q, u_{q+1}, \dots, u_l)].$$

It follows that $[P_x] \cdot [P_y] = \Phi(t_x t_y)$, and we are done. \square

5.2. Oddly-connected Coxeter systems. Define a Coxeter system (W, S) to be *oddly-connected* if for every pair of vertices s, t in its Coxeter diagram G , there is a walk in G of the form $(s = v_1, v_2, \dots, v_k = t)$ where the edge weight $m(v_i, v_{i+1})$ is odd for all $1 \leq i \leq k-1$. In this subsection, we discuss how the odd-weight edges affect the structure of the algebras J_C and J_s ($s \in S$).

We need some relatively heavy notation.

Definition 5.3. For any $s, t \in S$ such that $M = m(s, t)$ is odd:

- (1) We define

$$z(st) = sts \dots t$$

to be the alternating word of length $M-1$ that starts with s . Note that it necessarily ends with t now that M is odd.

- (2) We define maps $\lambda_s^t, \rho_t^s : J_C \rightarrow J_C$ by

$$\lambda_s^t(t_x) = t_{z(st)} t_x,$$

$$\rho_t^s(t_x) = t_x t_{z(st)},$$

and define the map $\phi_s^t : J_C \mapsto J_C$ by

$$\phi_s^t(t_x) = \rho_t^s \circ \lambda_s^t(t_x)$$

for all $x \in C$.

Remark 5.4. The notation above is set up in the following way. The letters λ and ρ indicate a map is multiplying its input by an element on the left and right, respectively. The subscripts and superscripts are to provide mnemonics for what the maps do on the reduced words indexing the basis elements of J_C : by Corollary 4.22, λ_s^t maps $J_{\Gamma_s^{-1}}$ to $J_{\Gamma_t^{-1}}$ and vanishes on $J_{\Gamma_h^{-1}}$ for any $h \in S \setminus \{s\}$. Similarly, ρ_t^s maps J_{Γ_s} to J_{Γ_t} and vanishes on J_{Γ_h} for any $h \in S \setminus \{s\}$.

Proposition 5.5. Let s, t be as in Definition 5.3. Then

- (1) $\rho_t^s \circ \lambda_s^t = \lambda_s^t \circ \rho_t^s$.
- (2) $\rho_t^s \circ \rho_t^s(t_x) = t_x$ for any $x \in \Gamma_s$, $\lambda_t^s \circ \lambda_s^t(t_x) = t_x$ for any $x \in \Gamma_s^{-1}$.
- (3) $\rho_t^s(t_x) \lambda_s^t(t_y) = t_x t_y$ for any $x \in \Gamma_s, y \in \Gamma_s^{-1}$.
- (4) The restriction of ϕ_s^t on J_s gives an isomorphism of based rings from J_s to J_t .

Proof. Part (1) holds since both sides of the equation sends t_x to $t_{z(st)} t_x t_{z(st)}$. Parts (2) and (3) are consequences of the truncated Clebsch-Gordan rule. By the rule,

$$t_{z(st)} t_{z(st)} = t_s,$$

therefore $\rho_t^s \circ \rho_t^s(t_x) = t_x t_s = t_x$ for any $x \in \Gamma_s$ and $\lambda_t^s \circ \lambda_s^t(t_x) = t_s t_x$ for any $x \in \Gamma_s^{-1}$; this proves (2). Meanwhile, $\rho_t^s(t_x) \lambda_s^t(t_y) = t_x t_{z(st)} t_{z(st)} t_y = t_x t_s t_y = t_x t_y$ for any $x \in \Gamma_s, y \in \Gamma_s^{-1}$; this proves (3).

For part (4), the fact that ϕ_s^t maps J_s to J_t follows from Remark 5.4. To see that is a (unit-preserving) algebra homomorphism, note that

$$\phi_s^t(t_s) = t_{z(st)} t_s t_{z(st)} = t_{z(st)} t_{z(st)} = t_t,$$

and for all $t_x, t_y \in J_s$,

$$\phi_s^t(t_x) \phi_s^t(t_y) = (\rho_t^s(\lambda_s^t(t_x))) \cdot (\lambda_s^t(\rho_t^s(t_y))) = \lambda_s^t(t_x) \cdot \rho_t^s(t_y) = \phi_s^t(t_x t_y)$$

by parts (1) and (3). We can similarly check ϕ_t^s is an algebra homomorphism from J_t to J_s . Finally, using calculations similar to those used for part (2), it is easy to check that ϕ_s^t and ϕ_t^s are mutual inverses, therefore ϕ_s^t is an algebra isomorphism.

It remains to check that the restriction is an isomorphism of based rings. In light of Proposition 3.15, this means checking that $\phi_s^t(t_{x-1}) = (\phi_s^t(t_x))^*$ for each $t_x \in J_s$, where $*$ is the linear map sending t_x to t_{x-1} for each $t_x \in J_s$. This holds because

$$\phi_s^t(t_{x-1}) = t_{z(st)}t_{x-1}t_{z(st)} = (t_{z(st)-1}t_x t_{z(st)-1})^* = (t_{z(st)}t_x t_{z(st)})^* = (\phi_s^t(t_x))^*,$$

where the second equality follows from the definition of $*$ and the fact that $t_x \mapsto t_{x-1}$ defines an anti-homomorphism in J (see Corollary 3.2). \square

Now we upgrade the definitions and propositions from a single edge to a walk.

Definition 5.6. For any walk $P = (u_1, \dots, u_l)$ in G where $m(u_k, u_{k+1})$ is odd for all $1 \leq k \leq l-1$, we define maps λ_P, ρ_P by

$$\begin{aligned}\lambda_P &= \lambda_{u_{l-1}}^{u_l} \circ \dots \circ \lambda_{u_2}^{u_3} \circ \lambda_{u_1}^{u_2}, \\ \rho_P &= \rho_{u_{l-1}}^{u_l} \circ \dots \circ \rho_{u_2}^{u_3} \circ \rho_{u_1}^{u_2},\end{aligned}$$

and define the map $\phi_P : J_C \rightarrow J_C$ by

$$\phi_P = \lambda_P \circ \rho_P.$$

Proposition 5.7. Let $P = (u_1, \dots, u_l)$ be as in Definition 5.6 Then

- (1) $\phi_P = \phi_{u_{l-1}}^{u_l} \circ \dots \circ \phi_{u_2}^{u_3} \circ \phi_{u_1}^{u_2}$.
- (2) $\rho_{P^{-1}} \circ \rho_P(t_x) = t_x$ for any $x \in \Gamma_{u_1}$, $\lambda_{P^{-1}} \circ \lambda_P(t_x) = t_x$ for any $x \in \Gamma_{u_l}^{-1}$.
- (3) $\rho_P(t_x)\lambda_P(t_y) = t_x t_y$ for any $x \in \Gamma_{u_1}, y \in \Gamma_{u_l}^{-1}$.
- (4) The restriction of ϕ_P gives an isomorphism of based rings from J_{u_1} to J_{u_l} .

Proof. Part (1) holds since each left multiplication $\lambda_{u_k}^{u_{k+1}}$ commutes with all right multiplications $\rho_{u_{k'}}^{u_{k'+1}}$. Part (2)-(4) can be proved by writing out each of the maps as a composition of $(l-1)$ appropriate maps corresponding to the $(l-1)$ edges of P and then repeatedly applying their counterparts in Proposition 5.7 on the composition components. In particular, part (4) holds since ϕ_P is a composition of isomorphisms of based rings is clearly another isomorphism of based rings. \square

We are almost ready to prove Theorem B:

Theorem B. Let (W, S) be an oddly-connected Coxeter system. Then

- (1) $J_s \cong J_t$ as based rings for all $s, t \in S$.
- (2) $J_C \cong \text{Mat}_{S \times S}(J_s)$ as based rings for all $s \in S$. In particular, J_C is Morita equivalent to J_s for all $s \in S$.

Here, for each fixed $s \in S$, the algebra $\text{Mat}_{S \times S}(J_s)$ is the matrix algebra of matrices with rows and columns indexed by S and with entries from J_s . For any $a, b \in S$ and $f \in J_s$, let $E_{a,b}(f)$ be the matrix in $\text{Mat}_{S \times S}(J_s)$ with f at the a -row, b -column and zeros elsewhere. We explain how $\text{Mat}_{S \times S}(J_s)$ is a based ring below.

Proposition 5.8. The ring $\text{Mat}_{S \times S}(J_s)$ is a based ring with basis $\{E_{a,b}(t_x) : a, b \in S, x \in \Gamma_s \cap \Gamma_s^{-1}\}$, with unit element $1 = \sum_{s \in S} E_{s,s}(t_s)$, and with its anti-involution induced by $E_{a,b}(t_x)^* = E_{b,a}(t_{x^{-1}})$.

Proof. Note that for any $a, b, c, d \in S$ and $f, g \in J_s$,

$$(15) \quad E_{a,b}(f)E_{c,d}(g) = \delta_{b,c}E_{a,d}(fg).$$

The fact that $\text{Mat}_{S \times S}(J_s)$ is a unital \mathbb{Z}_+ -ring with $1 = \sum_{s \in S} E_{s,s}(t_s)$ is then straightforward to check. Next, note that

$$(E_{a,b}(f)E_{c,d}(g))^* = 0 = (E_{c,d}(g))^*(E_{a,b}(f))^*$$

when $b \neq c$. When $b = c$,

$$(E_{a,b}(t_x)E_{c,d}(t_y))^* = (E_{a,d}((t_x t_y)))^* = E_{d,a}(t_{y^{-1}} t_{x^{-1}}) = (E_{c,d}(t_y))^* (E_{a,b}(t_x))^*$$

where, like in the proof of Proposition 5.5, the second equalities again follows from the fact that the map $t_x \mapsto t_{x^{-1}}$ induces an anti-homomorphism of J . The last two displayed equations imply that $*$ induces an anti-involution of $\text{Mat}_{S \times S}(J_s)$. Finally, note that $E_{u,u}(t_s)$ appears in $E_{a,b}(t_x)E_{c,d}(t_y) = \delta_{b,c}E_{a,d}(t_x t_y)$ for some $u \in S$ if and only if $b = c, a = d = u$ and $x = y^{-1}$ (for t_s appears in $t_x t_y$ if and only if $x = y^{-1}$; see Corollary 3.15). This proves that Equation (6) from Definition 3.12 holds, and we have completed all the necessary verifications. \square

Proof of Theorem. Part (1) follows from the last part of Proposition 5.7, since there is a walk $(s = u_1, u_2, \dots, u_l = t)$ in G that contains only odd-weight edges now that (W, S) is oddly-connected.

To prove (2), fix $s \in S$. For each $t \in S$, fix a walk $P_{st} = (s = u_1, \dots, u_l = t)$ and define $P_{ts} = P_{st}^{-1}$. Write λ_{st} for $\lambda_{P_{st}}$, and define $\rho_{st}, \lambda_{ts}, \rho_{ts}$ similarly. Consider the unique \mathbb{Z} -module map

$$\Psi : J_C \rightarrow \text{Mat}_{S \times S}(J_s)$$

defined as follows: for any $t_x \in J_C$, say $x \in \Gamma_a^{-1} \cap \Gamma_b$ for $a, b \in S$, let

$$\Psi(t_x) = E_{a,b}(\lambda_{as} \circ \rho_{bs}(t_x)).$$

We first show below that Ψ is an algebra isomorphism.

Let $t_x, t_y \in J_C$. Suppose $x \in \Gamma_a^{-1} \cap \Gamma_b$ and $y \in \Gamma_c^{-1} \cap \Gamma_d$ for $a, b, c, d \in S$. If $b \neq c$, then

$$\Psi(t_x)\Psi(t_y) = 0 = \Psi(t_x t_y),$$

where the first equality follows from Equation (15) and the second equality holds since $t_x t_y = 0$ by Corollary 4.22. If $b = c$, then

$$\begin{aligned} \Psi(t_x)\Psi(t_y) &= E_{a,b}(\lambda_{as} \circ \rho_{bs}(t_x)) \cdot E_{c,d}(\lambda_{cs} \circ \rho_{ds}(t_y)) \\ &= E_{a,d}([\lambda_{as} \circ \rho_{bs}(t_x)] \cdot [\lambda_{bs} \circ \rho_{ds}(t_y)]) \\ &= E_{a,d}((\lambda_{as} \circ \rho_{ds})[\rho_{bs}(t_x) \cdot \lambda_{bs}(t_y)]) \\ &= E_{a,d}((\lambda_{as} \circ \rho_{ds})[t_x t_y]) \\ &= \Psi(t_x t_y), \end{aligned}$$

where the second last equality holds by part (3) of Proposition 5.7. It follows that Ψ is an algebra homomorphism. Next, consider the map

$$\Psi' : \text{Mat}_{S \times S}(J_s) \rightarrow J_C$$

defined by

$$\Psi'(E_{a,b}(f)) = \lambda_{sa} \circ \rho_{sb}(f)$$

for all $a, b \in S$ and $f \in J_s$. Using Part (2) of Proposition 5.7, it is easy to check that Ψ and Ψ' are mutual inverses as maps of sets. It follows that Ψ is an algebra isomorphism. Finally, it is easy to compare Proposition 3.15 with Proposition 5.8 and check that Ψ is an isomorphism of based rings by direct computation. \square

Remark 5.9. The conclusions of the theorem fail in general when (W, S) is not oddly-connected. As a counter-example, consider based rings J_1 and J_2 arising from the Coxeter system in Example 4.27. By the truncated Clebsch-Gordan rule,

$$t_{212}t_{212} = t_2 = t_{232}t_{232},$$

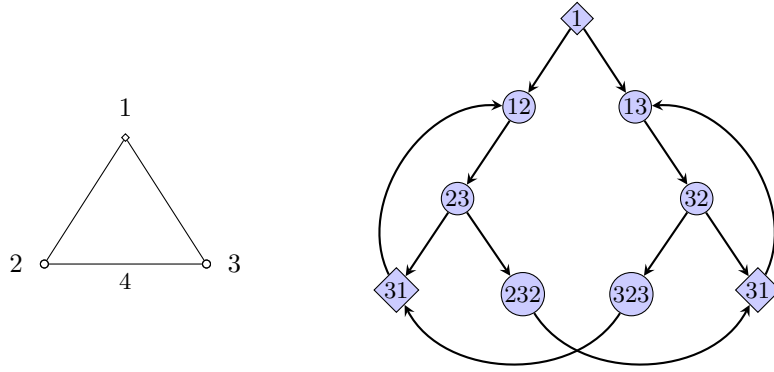
therefore J_2 contains at least two basis elements with multiplicative order 2. However, it is evident from Example 4.27 that t_{121} is the only basis element of order 2 in J_1 . This implies that J_1 and J_2 are not isomorphic as based rings. Moreover, Equation (15) implies that for any $s \in S$, the basis elements of $\text{Mat}_{S \times S}(J_s)$ of order

2 must be of the form $E_{u,u}(t_x)$ where $u \in S$ and t_x is a basis element of order 2 in J_s , so $\text{Mat}_{S \times S}(J_1)$ and $\text{Mat}_{S \times S}(J_2)$ have different numbers of basis elements of order 2 as well. It follows that Part (2) of the theorem also fails.

Remark 5.10. The isomorphism between J_s and J_t can be easily lifted to a tensor equivalence between their categorifications \mathcal{J}_s and \mathcal{J}_t , the subcategories of the category \mathcal{J} mentioned in the introduction that correspond to $\Gamma_s \cap \Gamma_s^{-1}$ and $\Gamma_t \cap \Gamma_t^{-1}$.

Let us end the section by revisiting an earlier example.

Example 5.11. Let (W, S) be the Coxeter system from Example 4.8, whose Coxeter diagram and subregular graph are shown again below.



Clearly, (W, S) is *oddly-connected*, hence $J_3 \cong J_2 \cong J_1$ and $J_C \cong \text{Mat}_{3 \times 3}(J_1)$ by Theorem 1. Let us study J_1 . Recall that elements of $\Gamma_1 \cap \Gamma_1^{-1}$ correspond to walks on the subregular graph that start with the top vertex and end with either the bottom-left or bottom-right vertex. Observe that all such walks can be obtained by concatenating the walks corresponding to the elements $x = 1231, y = 1321, z = 12321, w = 13231$. This means that any reduced word in $\Gamma_1 \cap \Gamma_1^{-1}$ can be written as glued products of x, y, z, w , which implies that t_x, t_y, t_z, t_w generate J_1 by Theorem F and Proposition 4.25. Computing the products of these elements reveals that J_1 can be described as the algebra generated by t_x, t_y, t_z, t_w subject to the following six relations:

$$t_x t_y = 1 + t_z, t_y t_x = 1 + t_w, t_x t_w = t_x = t_z t_x, t_y t_z = t_y = t_w t_y, t_w^2 = 1 = t_z^2.$$

The first two of the relations show that $t_z = t_x t_y - 1, t_w = t_y t_x - 1$, whence the other four relations can be expressed in terms of only t_x and t_y . Easy calculations then show that J_1 can be presented as the algebra generated by t_x, t_y subject to only the following two relations:

$$t_x t_y t_x = 2t_x, t_y t_x t_y = 2t_y.$$

Finally, via the change of variables $X := t_x/2, Y := t_y$, we see that

$$J_1 = \langle X, Y \rangle / \langle XYX = X, YXY = Y \rangle.$$

A simple presentation like this is helpful for studying representations of J_1 and hence J_2, J_3 and J_C .

5.3. Fusion J_s . In this subsection, we describe all fusion rings appearing in the form J_s from a Coxeter system. Recall from Definition 3.13 that a fusion ring is a unital based ring of finite rank, so the algebra J_s is a fusion ring if and only if $\Gamma_s \cap \Gamma_s^{-1}$ is finite. It is easy to describe when this happens using Coxeter diagrams.

Proposition 5.12. *Let (W, S) be an irreducible Coxeter system. Then $\Gamma_s \cap \Gamma_s^{-1}$ is finite for some $s \in S$ if and only if $\Gamma_s \cap \Gamma_s^{-1}$ is finite for all $s \in S$. Moreover, both of these conditions are met if and only if the Coxeter graph G of (W, S) is a tree, no edge of G has weight ∞ , and at most one edge of G has weight greater than 3.*

Proof. Since (W, S) is irreducible, G is connected. The condition that G is a tree is then equivalent to the condition that G contains no cycle. Let D be the subregular graph of (W, S) . We need to show that one can find infinitely many s -walks on D for some $s \in S$ exactly when G contains a cycle or more than one edge of weight 3, exactly when we can find infinitely many s -walks on D for all $s \in S$. This is a routine and straightforward graph theory problem, and we omit the details. \square

We can now deduce Theorem C.

Theorem C. *Let (W, S) be a Coxeter system, and let $s \in S$. Suppose J_s is a fusion ring for some $s \in S$. Then there exists a dihedral Coxeter system (W', S') such that $J_s \cong J_{s'}$ as based rings for either $s' \in S$.*

Proof. Let G be the Coxeter diagram of (W, S) , and suppose J_s is a fusion ring for some $s \in S$. Then $\Gamma_s \cap \Gamma_s^{-1}$ is finite, hence G must be as described in the previous proposition, that is, either G is a tree and (W, S) is simply-laced, or G is a tree and there exists a unique pair $a, b \in S$ such that $m(a, b) > 3$.

In the first case where (W, S) is simply-laced, J_s is group algebra of the fundamental group $\Pi_s(G)$ by Theorem 1, and the group is trivial since G is a tree. This means J_s is isomorphic to a ring of the form $J_{s'}$ associated with the dihedral system (W', S') with $S' = \{s', t'\}$ and $m(s', t) = 3$. In the second case, let $m(a, b) = M$. By the description of G , there must be a walk P in G from s to either a or b such all the edges in the walk must have weight 3, so Part (4) of Proposition 5.7 implies that J_s is isomorphic to either J_a or J_b as based rings. Without loss of generality, suppose $J_s \cong J_a$. We claim that $\Gamma_a \cap \Gamma_a^{-1}$ contains exactly the elements $a, aba, \dots, ab \dots a$ where the reduced words alternate in a, b and contains less than M letters. This would mean that J_s is isomorphic as a based ring to the fusion ring $J_{s'}$ associated with the dihedral system (W', S') with $S' = \{s', t'\}$ where $m(s', t') = M$.

It remains to prove the claim. Recall that any element $x = s_1 \dots s_k \in \Gamma_a \cap \Gamma_a^{-1}$ corresponds to a walk $(a = s_1, s_2, \dots, s_k = a)$ on G . Since G is a tree, the walk must be the concatenation of walks P_{at} that start with a , traverse to a vertex $t \in S$ via the unique path from a to t , and then come back via the inverse path to a , i.e., $P_{at} = (a = s_1, \dots, s_{k-1}, s_k = t, s_{k-1}, \dots, s_1)$. The spur (s_{k-1}, t, s_k) in the walk means $s_{k-1}ts_{k-1}$ appears in x , so $m(t, s_{k-1}) > 3$ by Proposition 4.5, hence t must be a or b . The claim follows. \square

Remark 5.13. Recall from §4.4 that any algebra of the form J_s arising from a dihedral Coxeter system is isomorphic to the odd part $\text{Ver}_M^{\text{odd}}$ of a Verlinde algebra, where $M \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. Thus, the theorem means that any fusion ring of the form J_s arising from any Coxeter system (W, S) is isomorphic to $\text{Ver}_M^{\text{odd}}$ for some M as well. Moreover, the proof of the theorem reveals that M can be described simply as the largest edge weight in the Coxeter diagram of (W, S) .

6. FREE FUSION RINGS

We focus on certain Coxeter systems (W, S) whose Coxeter diagrams involve edges of weight ∞ in this section. We show that for suitable choice of $s \in S$, J_s is isomorphic to a *free fusion ring*.

6.1. **Background.** Free fusion rings are defined as follows.

Definition 6.1 ([Rau12]). A *fusion set* is a set A equipped with an involution $\bar{\cdot} : A \rightarrow A$ and a *fusion map* $\circ : A \times A \rightarrow A \cup \emptyset$. Given any fusion set $(A, \bar{\cdot}, \circ)$, we extend the operations $\bar{\cdot}$ and \circ to the free monoid $\langle A \rangle$ as follows:

$$\overline{a_1 \cdots a_k} = \bar{a}_k \cdots \bar{a}_1,$$

$$(a_1 \cdots a_k) \circ (b_1 \cdots b_l) = a_1 \cdots a_{k-1} (a_k \circ b_1) b_2 \cdots b_l,$$

where the right side of the last equation is taken to be \emptyset whenever $k = 0, l = 0$ or $a_k \circ b_1 = \emptyset$. We then define the *free fusion ring* associated with the fusion set $(A, \bar{\cdot}, \circ)$ to be the free abelian group $R = \mathbb{Z}\langle A \rangle$ on $\langle A \rangle$, with multiplication $\cdot : R \times R \rightarrow R$ given by

$$(16) \quad v \cdot w = \sum_{v=xy, w=\bar{y}z} xz + x \circ z$$

for all $v, w \in \langle A \rangle$, where xz means the juxtaposition of x and z .

It is well known that \cdot is associative (see [Rau12]). It is also easy to check that R is always a unital based ring with its basis given by the free monoid $\langle A \rangle$, with unit given by the empty word, and with its anti-involution $*$: $\langle A \rangle \rightarrow \langle A \rangle$ given by the map $\bar{\cdot}$.

Free fusion rings were introduced in [BV09] to capture the tensor rules in certain semisimple tensor categories arising from the theory of operator algebras. More specifically, the categories are categories of representations of *compact quantum groups*, and their Grothendieck rings fit the axiomatization of free fusion rings in Definition 6.1. In [Fre14], A. Freslon classified all free fusion rings arising as the Grothendieck rings of compact quantum groups in terms of their underlying fusion sets. Further, while a free fusion ring may appear as the Grothendieck ring of multiple non-isomorphic compact quantum groups, Freslon described a canonical way to associate a *partition quantum group*—a special type of compact quantum group—to any free fusion ring arising from a compact quantum group. These special quantum groups correspond via a type of Schur-Weyl duality to *categories of non-crossing partitions*, which can in turn be used to study the representations of the quantum groups.

All the free fusion rings appearing as J_s in our examples fit in the classification of [Fre14]. In each of our examples, we will identify the associated partition quantum group \mathbb{G} . The fact that J_s is connected to \mathbb{G} is intriguing, and it would be interesting to see how the categorification of J_s arising from Soergel bimodules connects to the representations of \mathbb{G} on the categorical level.

6.2. **Example 1:** O_N^+ . One of the simplest fusion set is the singleton set $A = \{a\}$ with identity as its involution and with fusion map $a \circ a = \emptyset$. The associated free fusion ring is $R = \oplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Z}a^n$, where

$$a^k \cdot a^l = a^{k+l} + a^{k+l-2} + \cdots + a^{|k-l|}$$

by Equation 16. The partition quantum group associated to R is the *free orthogonal quantum group* O_N^+ , and its corresponding category of partitions is that of all noncrossing *pairings* ([?]).

Let (W, S) be the infinite dihedral system with $S = \{1, 2\}$ and $W = I_2(\infty)$, the infinite dihedral group. We claim that J_1 is isomorphic to R as based rings. To see this, recall from the discussion following Definition 4.19 that J_s is the \mathbb{Z} -span of basis elements t_{1_n} , where n is odd and $1_n = 121 \cdots 1$ alternates in 1, 2 and

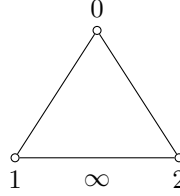
has length n . For $m = 2k + 1$ and $n = 2l + 1$ for some $k, l \geq 1$, the truncated Clebsch-Gordan rule implies that

$$t_{1_m} \cdot t_{1_n} = t_{1_{2k+1}} t_{1_{2l+1}} = t_{1_{2(k+l)+1}} + t_{1_{2(k+l-1)+1}} + \cdots + t_{1_{2|k-l|+1}}.$$

It follows that $R \cong J_1$ as based rings via the unique \mathbb{Z} -module map with $a^k \mapsto t_{1_{2k+1}}$ for all $k \in \mathbb{Z}_{\geq 0}$. Similarly, $R \cong J_2$ as based rings.

6.3. Example 2: U_N^+ . In this subsection we consider the free fusion ring R arising from the fusion set $A = \{a, b\}$ with $\bar{a} = b$ and $a \circ a = a \circ b = b \circ a = b \circ a = \emptyset$. The partition quantum group associated to R is the *free unitary quantum group* U_N^+ . In the language of [Fre14], this quantum group corresponds to the category of \mathcal{A} -colored noncrossing partitions where \mathcal{A} is a *color set* containing two colors *inverse* to each other.

Consider the Coxeter system (W, S) with the following Coxeter diagram.



Theorem D. *We have an isomorphism $R \cong J_0$ of based rings.*

Our strategy to prove Theorem 6.3 is to describe a bijection between the free monoid $\langle A \rangle$ and the set $\Gamma_0 \cap \Gamma_0^{-1}$, use it to define a \mathbb{Z} -module isomorphism from R to J_0 , then show that it is an isomorphism of based rings. To establish the bijection, recall from the discussion after Definition 5.1 that any element $x \in \Gamma_0 \cap \Gamma_0^{-1}$ corresponds to a unique walk P_x on the graph G . We may encode x by a word in $\langle A \rangle$ in the following way: imagine traversing the walk P_x , write down an “ a ” every time an edge in the walk goes from 1 to 2, a “ b ” every time an edge goes from 2 to 1, and write nothing down otherwise. Call the resulting word w_x . For example, the element $x = 012120120$ corresponds to the word $w_x = abaa$. Note that w_x records all parts of P_x that travel along the edge $\{1, 2\}$, but “ignores” the parts that involve the edges containing 0.

We claim that the map $\varphi : \Gamma_0 \cap \Gamma_0^{-1} \rightarrow \langle A \rangle, x \mapsto w_x$ gives our desired bijection. To see that φ is injective, note that by Proposition 4.5, the elements of $\Gamma_0 \cap \Gamma_0^{-1}$ correspond to walks on G that start and end with 0 but contain no spurs involving 0. The latter condition means that the parts of the walk P_x that is “ignored” in w_x , i.e., the parts involving the edges $\{0, 1\}$ or $\{0, 2\}$, can be recovered from w_x . More precisely, given any word $w = w_x$ for some $x \in \Gamma_0 \cap \Gamma_0^{-1}$, we may read the letters of w from left to right and write down P_x using the following principles:

- (1) The empty word $w = \emptyset$ corresponds to the element $0 \in \Gamma_0 \cap \Gamma_0^{-1}$, for P_x involves the edge $\{1, 2\}$ for any other element of the intersection.
- (2) The only way w can start with a and not b is for P_x to start with 0, immediately travel to 1, then travel from 1 to 2, so P_x must start with $(0, 1, 2)$ if w starts with a . Similarly, P_x starts with $(0, 2, 1)$ if w starts with b .
- (3) If the last letter we have read from w is an “ a ”, the last vertex we have recovered in the sequence for P_x must be 2.
 - (a) If this “ a ” is the last letter of w , P_x must involve no more traversals of the edge $\{1, 2\}$ and hence immediately return to 2 from 0, so adding one more 0 to the current sequence returns P_x .

-
- (b) If the “a” is followed by another “a”, the next traversal of $\{1, 2\}$ in P_x after the sequence already written down must be from 1 to 2 again. This forces P_x to travel to 0 next, and to avoid a spur it must go on to 1, then to 2, so we add $(0, 1, 2)$ to the sequence for P_x . If the “a” is followed by a “b”, P_x must next immediately travel to 1 and we add 1 to the sequence, for otherwise P_x would have to travel along the cycle $2 \rightarrow 0 \rightarrow 1 \rightarrow 2$ as we just described and the “a” would be followed by another “a”.
- (4) If the last letter we have read from w is an “b”, the last vertex we have recovered in the sequence for P_x must be 1. The method to recover more of P_x from the rest of w is similar to the one described in (3).

To illustrate the recovery of w_x from x , suppose we know $abaa = w_x$ for some $x \in \Gamma_0 \cap \Gamma_0^{-1}$, we would get $P_x = (0, 1, 2, 1, 2, 0, 1, 2, 0)$ by successively writing down $(0, 1, 2)$, (1) , (2) , $(0, 1, 2)$ and (0) , so $x = 012120120$. Indeed, note that we may run the process for any word w in $\langle A \rangle$ to get an element in $\Gamma_0 \cap \Gamma_0^{-1}$. This gives us a map $\phi : \langle A \rangle \rightarrow \Gamma_0 \cap \Gamma_0^{-1}$ that is a mutual inverse to φ and ϕ , so both φ and ϕ are bijective.

We can now prove Theorem D. We present an inductive proof that can be easily adapted to prove Theorem E later.

Proof of Theorem D. Let $\Phi : R \rightarrow J_0$ be the \mathbb{Z} -module homomorphism defined by

$$\Phi(w) = t_{\phi(w)}.$$

Since ϕ is a bijection, this is an isomorphism of \mathbb{Z} -modules. We will show that Φ is an algebra isomorphism by showing that

$$(17) \quad \Phi(v)\Phi(w) = \Phi(v \cdot w)$$

for all $v, w \in \langle A \rangle$. Note that this is true if v or w is empty, since then $t_v = t_0$ or $t_w = t_0$, which is the identity of J_0 by Corollary 4.12.

Now, assume neither v nor w is empty. We prove Equation (17) by induction on the length $l(v)$ of v , i.e., on the number of letters in v . For the base case, suppose $l(v) = 1$ so that $v = a$ or $v = b$. If $v = a$, then $\phi(a) = 0120$. There are two cases:

- (1) Case 1: w starts with a .

Then $\phi(w)$ has the form $\phi(w) = 012 \dots$, so

$$\Phi(v)\Phi(w) = t_{0120}t_{012\dots} = t_{0120*012\dots} = t_{012012\dots} = t_{\phi(aw)}$$

by Proposition 4.25. Meanwhile, since $\bar{a} \neq a$ and $a \circ a = \emptyset$ in A ,

$$v \cdot w = aw$$

in R , therefore $\Phi(v \cdot w) = t_{\phi(aw)}$ as well. Equation (17) follows.

- (2) Case 2: w starts with b .

In this case, suppose the longest alternating subword $bab \dots$ appearing in the beginning of w has length k , and write $w = bw'$. Then $\phi(w)$ takes the form $\phi(w) = 0212 \dots$, its first dihedral segment is 02 , the second is $(2, 1)_{k+1}$, so that $\phi(w) = 02 * (2, 1)_{k+1} * x$ where x is the glued product of all the remaining dihedral segments. Direct computation using Theorem F and propositions 4.25 and 4.17 then yields

$$\begin{aligned} \Phi(v)\Phi(w) &= t_{01}[t_{(1,2)_{k+2}} + t_{(1,2)_k}]t_x \\ &= t_{01*(1,2)_{k+2}*x} + t_{01*(1,2)_k*x} \\ &= t_{\phi(w)} + t_{\phi(w')}. \end{aligned}$$

Meanwhile, since $\bar{a} = b$ and $a \circ b = \emptyset$ in A ,

$$v \cdot w = a \cdot bab \dots = abab \dots + ab \dots = w + w'$$

in R , therefore $\Phi(v \cdot w) = t_{\phi(w)} + t_{\phi(w')}$ as well. Equation (17) follows.

The proof for the case $l(v) = 1$ and $v = b$ is similar.

For the inductive step of our proof, assume Equation (17) holds whenever v is nonempty and $l(v) < L$ for some $L \in \mathbb{N}$, and suppose $l(v) = L$. Let $\alpha \in A$ be the first letter of v , and write $v = \alpha v'$. Then $l(v') < L$, and by (16),

$$\alpha \cdot v' = v + \sum_{u \in U} u$$

where U is a subset of $\langle A \rangle$ where all words have length smaller than L . Using the inductive hypothesis on α, v', u and the \mathbb{Z} -linearity of Φ , we have

$$\begin{aligned} \Phi(v)\Phi(w) &= \Phi\left(\alpha \cdot v' - \sum_{u \in U} u\right)\Phi(w) \\ &= \Phi(\alpha)\Phi(v')\Phi(w) - \sum_{u \in U} \Phi(u)\Phi(w) \\ &= \Phi(\alpha)\Phi(v' \cdot w) - \Phi\left(\sum_{u \in U} u \cdot w\right) \end{aligned}$$

Here, the element $v' \cdot w$ may be a linear combination of multiple words in R , but applying the inductive hypothesis on α still yields

$$\Phi(\alpha)\Phi(v' \cdot w) = \Phi(\alpha \cdot (v' \cdot w))$$

by the \mathbb{Z} -linearity of Φ and \cdot . Consequently,

$$\begin{aligned} \Phi(v)\Phi(w) &= \Phi(\alpha \cdot (v' \cdot w)) - \Phi\left(\sum_{u \in U} u \cdot w\right) \\ &= \Phi\left((\alpha \cdot v') \cdot w - \sum_{u \in U} u \cdot w\right) \\ &= \Phi\left(\left[(\alpha \cdot v') - \sum_{u \in U} u\right] \cdot w\right) \\ &= \Phi(v \cdot w). \end{aligned}$$

by the associativity of \cdot and the \mathbb{Z} -linearity of Φ and \cdot . This completes the proof that Φ is an algebra isomorphism.

The fact that Φ is in addition an isomorphism of based rings is straightforward to check. In particular, observe that $\phi(\bar{w}) = \phi(w)^{-1}$ so that $\Phi(\bar{w}) = t_{\phi(\bar{w})} = t_{\phi(w)^{-1}} = (\Phi(w))^*$, therefore Φ is compatible with the respective involutions in R and J_0 . We omit the details of the other necessary verifications. \square

6.4. Example 3: $Z_N^+(\{e\}, n-1)$. In this subsection, we consider an infinite family of fusion rings $\{R_n : n \in \mathbb{Z}_{\geq 2}\}$, where each R_n arises from the fusion set

$$A_n = \{e_{ij} : i, j \in [n]\}$$

with $\bar{e}_{ij} = e_{ji}$ for all $i, j \in [n]$ and

$$e_{ij} \circ e_{kl} = \begin{cases} e_{il} & \text{if } j = k \\ \emptyset & \text{if } j \neq k \end{cases}$$

for all $i, j, k, l \in [n]$. We may think of the fusion set as the usual matrix units for $n \times n$ matrices and think of the fusion map as an analog of matrix multiplication, with the fusion product being \emptyset whenever the matrix product is 0. In the notation of [Fre14], the partition quantum group corresponding to R_n is denoted by

$Z_N^+(\{e\}, n-1)$, which equals the *amalgamated free product* of $(n-1)$ copies of \tilde{H}_N^+ amalgamated along S_N^+ , where S_N^+ stands for the *free symmetric group*, H_N^+ stands for the *free hyperoctohedral group*, and \tilde{H}_N^+ stands for the *free complexification* of H_N^+ . In particular, $R_2 = \tilde{H}_N^+$.

For $n \in \mathbb{Z}_{\geq 2}$, let (W_n, S_n) be the Coxeter system where $S_n = \{0, 1, 2, \dots, n\}$, $m(0, i) = \infty$ for all $i \in [n]$, $m(i, i+1) = 3$ for all $i \in [n-1]$, and $m(i, j) = 2$ otherwise. The Coxeter diagrams G_n of (W_n, S_n) are shown in Figure 6.4.

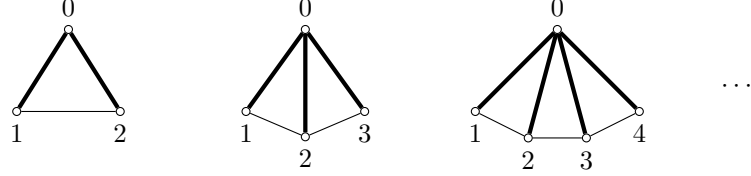


FIGURE 2. The Coxeter diagrams of (W_n, S_n) . The thick edges have weight ∞ ; the remaining edges have weight 3.

Let $J_0^{(n)}$ denote the subring J_0 of the subregular J -ring of (W_n, S_n) .

Theorem E. *For each $n \in \mathbb{Z}_{\geq 2}$, $R_n \cong J_0^{(n)}$ as based rings.*

For each $n \geq 2$, the strategy to prove the isomorphism $R_n \cong J_0^{(n)}$ is similar to the one used for Theorem D. That is, we will first describe a bijection $\phi : \langle A_n \rangle \rightarrow \Gamma_0 \cap \Gamma_0^{-1}$, then show that the \mathbb{Z} -module map $\Phi : R_n \rightarrow J_0^0$ given by $\Phi(w) = t_{\phi(w)}$ is an isomorphism of based rings.

To describe ϕ , note that for $i, j \in [n]$, there is a unique shortest walk P_{ij} from i to j on the “bottom part” of G_n , i.e., on the subgraph of G_n induced by the vertex subset $[n]$. Define $\phi(e_{ij})$ to be element in $\Gamma_0 \cap \Gamma_0^{-1}$ corresponding to the walk on G that starts from 0, travels to i , traverses to j along the path P_{ij} , then returns to 0. For example, when $n = 4$, $\phi(e_{24}) = 02340$, $\phi(e_{43}) = 0430$, $\phi(e_{44}) = 040$. Next, for any word w in $\langle A_n \rangle$, define $\phi(w)$ to be the glued product of the ϕ -images of its letters. For example, $\phi(e_{24}e_{43}e_{44}e_{44}) = 023404304040$.

It is clear that ϕ is a bijection, with inverse φ given as follows: for any $x \in \Gamma_0 \cap \Gamma_0^{-1}$, write x as the glued product of subwords that start and end with 0 but do not contain 0 otherwise; each such subword must be of the form $\phi(e_{ij})$. We define $\varphi(x)$ to be the concatenation of these letters. For example, $\varphi(0230404) = e_{23}e_{44}e_{44}$ since $0230404 = (0230) * (040) * (040)$.

Before we prove Theorem 6.4, let us record one useful lemma:

Lemma 6.2. *Let $x_{ij} = i \dots j$ be the element in C corresponding to the walk P_{ij} for all $i, j \in [n]$. Then $t_{x_{ij}}t_{x_{jk}} = t_{x_{ik}}$ for all $i, j, k \in [n]$.*

Proof. This follows by carefully considering the possible relationships between i, j, k and repeatedly using Proposition 4.17 to directly compute $t_{x_{ij}}t_{x_{jk}}$ in each case. Alternatively, notice that the simple reflection 0 is not involved in x_{ij} for any $i, j \in [n]$, hence the computation of $t_{x_{ij}}t_{x_{jk}}$ can be done in the subregular J -ring of the Coxeter system with the “bottom part” of G_n as its diagram. This system is simply-laced, so the result follows immediately from Theorem 1. \square

Proof of Theorem E. Let $n \geq 2$, and let ϕ and Φ be as above. As in the proof of Theorem 6.3, we show that Φ is an algebra isomorphism by checking that

$$(18) \quad \Phi(v)\Phi(w) = \Phi(v \cdot w)$$

for all $v, w \in \langle A_n \rangle$. Once again, we may assume that both v and w are non-empty again use induction on the length $l(v)$ of v . The inductive step of the proof will be

identical with the one for Theorem D. For the base case where $l(v) = 1$, suppose $v = e_{ij}$ for some $i, j \in [n]$. There are two cases.

- (1) Case 1: w starts with a letter $e_{j'k}$ where $j' \neq j$.

Then $\phi(v)$ and $\phi(w)$ take the form $\phi(v) = \cdots j0$, $\phi(w) = 0j' \cdots$, so

$$\Phi(v)\Phi(w) = t_{\cdots j0}t_{0j' \cdots} = t_{\cdots j0*0j' \cdots} = t_{\phi(e_{ij})*\phi(w)} = t_{\phi(e_{ij}w)}$$

by Proposition 4.25. Meanwhile, since $\bar{e}_{ij} \neq e_{j'k}$ and $e_{ij} \circ e_{j'k} = \emptyset$ in A_n ,

$$v \cdot w = e_{ij}w$$

in R , therefore $\Phi(v \cdot w) = t_{\phi(e_{ij}w)}$ as well. Equation (18) follows.

- (2) Case 2: w starts with e_{jk} for some $k \in [n]$.

Write $w = e_{jk}w'$. We need to carefully consider four subcases, according to how they affect the dihedral segments of $\phi(v)$ and $\phi(w)$.

- (a) $i = j = k$. Then $v = e_{jj}$, $\phi(v) = 0j0 = (0, j)_3$, and w starts with $e_{jj} \cdots$, hence $\phi(w)$ starts with $0j0 \cdots$. Suppose the first dihedral segment of $\phi(w)$ is $(0, j)_L$, and write $\phi(w) = (0, j)_L * x$. Then Theorem F and propositions 4.25 and 4.17 yield

$$\begin{aligned} \Phi(v)\Phi(w) &= t_{(0,j)_3}t_{(0,j)_L}t_x \\ &= t_{(0,j)_{L+2}*x} + t_{(0,j)_L*x} + t_{(0,j)_{L-2}*x} \\ &= t_{\phi(e_{jj}w)} + t_{\phi(w)} + t_{\phi(w')}, \end{aligned}$$

while

$$v \cdot w = e_{jj} \cdot e_{jj}w' = e_{jj}e_{jj}w' + e_{jj}w' + w' = e_{jj}w + w + w'$$

since $\bar{e}_{jj} = e_{jj}$ and $e_{jj} \circ e_{jj} = e_{jj}$. It follows that Equation (18) holds.

- (b) $i = j$, but $j \neq k$. In this case, $v = e_{jj}$, $\phi(v) = (0, j)_3$ as in (a), while $\phi(w) = 0j * x$ for some reduced word x which starts with j but not $j0$. We have

$$\Phi(v)\Phi(w) = t_{0j0}t_{j0}t_x = t_{0j0j*x} + t_{0j*x} = t_{\phi(e_{jj}w)} + t_{\phi(w)},$$

while

$$v \cdot w = e_{jj} \cdot e_{jk}w' = e_{jj}e_{jk}w' + e_{jk}w' = e_{jj}w + w$$

since $\bar{e}_{jj} \neq e_{jk}$ and $e_{jj} \circ e_{jk} = e_{jk}$. This implies Equation (18).

- (c) $i \neq j$, but $j = k$. In this case, $v = e_{ij}$ and $\phi(v) = y * j0$ for some reduced word y which ends in j but not $0j$, and $\phi(w)$ can be written as $\phi(w) = (0, j)_L * x$ as in (a). We have

$$\begin{aligned} \Phi(v)\Phi(w) &= t_y t_{j0} t_{(0,j)_L} t_x \\ &= t_{y*(j,0)_{L+1}*x} + t_{y*(j,0)_{L-1}*x} \\ &= t_{\phi(e_{ij}w)} + t_{\phi(e_{ij}w')}, \end{aligned}$$

while

$$v \cdot w = e_{ij} \cdot e_{jj}w' = e_{ij}w + e_{ij}w'$$

since $\bar{e}_{ij} \neq e_{jj}$ and $e_{ij} \circ e_{jj} = e_{ij}$. This implies Equation (18).

- (d) $i \neq j$, and $j \neq k$. In this case, $\phi(v) = 0i * x_{ij} * j0$ (recall the definition of x_{ij} from Lemma 6.2), and $\phi(w) = 0j * x_{jk} * x$ for some x which starts with $k0$. We have

$$\begin{aligned} \Phi(v)\Phi(w) &= t_{0i}t_{x_{ij}}t_{j0}t_{0j}t_{x_{jk}}t_x \\ &= t_{0i}t_{x_{ij}}t_{j0j}t_{x_{jk}}t_x + t_{0i}t_{x_{ij}}t_{jt_{x_{jk}}}t_x \\ &= t_{0i*x_{ij}*j0j*x_{jk}*x} + t_{0i}t_{x_{ij}}t_{x_{jk}}t_x \\ &= t_{\phi(e_{ij}w)} + t_{0i}t_{x_{ik}}t_x, \end{aligned}$$

where the fact $t_{x_{ij}}t_{x_{jk}} = t_{x_{ik}}$ comes from Lemma 6.2. Now, if $i \neq k$, $t_{0i}t_{x_{ik}}t_x = t_{0i*x_{ik}*x} = t_{\phi(e_{ik}w')}$, so

$$\Phi(v)\Phi(w) = t_{\phi(e_{ij}w)} + t_{\phi(e_{ik}w')}.$$

If $i = k$, note that $t_{0i}t_{ik}t_x = t_{0k}t_kt_x = t_{0k}t_x$. Suppose the first dihedral segment of x is $(k, 0)_{L'}$ for some $L' \geq 2$, and write $x = (k, 0)_{L'} * x'$. Then $t_{0k}t_x = t_{0k}t_{(k,0)_{L'+1}}t_{x'} = t_{(0,k)_{L'+1}*x'} + t_{(0,k)_{L'-1}*x'} = t_{\phi(e_{kk}w')+\phi(w')}$, so

$$\Phi(v)\Phi(w) = t_{\phi(e_{ij}w)} + t_{\phi(e_{ik}w')} + t_{\phi(w')}.$$

In either case, Equation (18) holds again, because

$$v \cdot w = e_{ij} \cdot e_{jk}w' = e_{ij}e_{jk}w' + e_{ik}w' + \delta_{ik}e_{w'} = e_{ij}w + e_{ik}w' + \delta_{ik}e_{w'}$$

now that $\bar{e}_{ij} = e_{ji}$ and $e_{ij} \circ e_{jk} = e_{ik}$.

We have now proved Φ is an algebra isomorphism. Just as in Theorem D, the fact that Φ is in addition an isomorphism of based rings is again easy to check, and we omit the details. \square

REFERENCES

- [AH74] A. V. Aho and J. E. Hopcroft. *The Design and Analysis of Computer Algorithms*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1st edition, 1974.
- [BB05] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.
- [Bez04] R. Bezrukavnikov. On tensor categories attached to cells in affine Weyl groups. In *Representation theory of algebraic groups and quantum groups*, volume 40 of *Adv. Stud. Pure Math.*, pages 69–90. Math. Soc. Japan, Tokyo, 2004.
- [BFO09] R. Bezrukavnikov, M. Finkelberg, and V. Ostrik. On tensor categories attached to cells in affine Weyl groups. III. *Israel J. Math.*, 170:207–234, 2009.
- [BK01] B. Bakalov and A. Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.
- [BO04] R. Bezrukavnikov and V. Ostrik. On tensor categories attached to cells in affine Weyl groups. II. In *Representation theory of algebraic groups and quantum groups*, volume 40 of *Adv. Stud. Pure Math.*, pages 101–119. Math. Soc. Japan, Tokyo, 2004.
- [BV09] T. Banica and R. Vergnioux. Fusion rules for quantum reflection groups. *J. Noncommut. Geom.*, 3(3):327–359, 2009.
- [CdSW99] A. Cannas da Silva and A. Weinstein. *Geometric models for noncommutative algebras*, volume 10 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.
- [CIK71] C. W. Curtis, N. Iwahori, and R. Kilmoyer. Hecke algebras and characters of parabolic type of finite groups with (B, N) -pairs. *Inst. Hautes Études Sci. Publ. Math.*, (40):81–116, 1971.
- [CR90] C. W. Curtis and I. Reiner. *Methods of representation theory. Vol. I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1990. With applications to finite groups and orders, Reprint of the 1981 original, A Wiley-Interscience Publication.
- [Cur87] C.W. Curtis. The Hecke algebra of a finite Coxeter group. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47 of *Proc. Sympos. Pure Math.*, pages 51–60. Amer. Math. Soc., Providence, RI, 1987.
- [Dev16] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 7.2.)*. <http://www.sagemath.org>, 2016.
- [DJ86] R. Dipper and G. James. Representations of Hecke algebras of general linear groups. *Proc. London Math. Soc.* (3), 52(1):20–52, 1986.
- [EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.
- [EK95] P. Etingof and M. Khovanov. Representations of tensor categories and Dynkin diagrams. *Internat. Math. Res. Notices*, (5):235–247, 1995.
- [EW14] B. Elias and G. Williamson. The Hodge theory of Soergel bimodules. *Ann. of Math.* (2), 180(3):1089–1136, 2014.

-
- [Fre14] A. Freslon. Fusion (semi)rings arising from quantum groups. *J. Algebra*, 417:161–197, 2014.
- [Gec98] M. Geck. Kazhdan-Lusztig cells and decomposition numbers. *Represent. Theory*, 2:264–277 (electronic), 1998.
- [GP00] M. Geck and G. Pfeiffer. *Characters of finite Coxeter groups and Iwahori-Hecke algebras*, volume 21 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 2000.
- [Hum90] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [Kas95] C. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [KL79] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.
- [Lus83] G. Lusztig. Some examples of square integrable representations of semisimple p -adic groups. *Trans. Amer. Math. Soc.*, 277(2):623–653, 1983.
- [Lus84] G. Lusztig. *Characters of reductive groups over a finite field*, volume 107 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1984.
- [Lus87a] G. Lusztig. Cells in affine Weyl groups. II. *J. Algebra*, 109(2):536–548, 1987.
- [Lus87b] G. Lusztig. Cells in affine Weyl groups. III. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(2):223–243, 1987.
- [Lus87c] G. Lusztig. Leading coefficients of character values of Hecke algebras. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, volume 47 of *Proc. Sympos. Pure Math.*, pages 235–262. Amer. Math. Soc., Providence, RI, 1987.
- [Lus89] G. Lusztig. Cells in affine Weyl groups. IV. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 36(2):297–328, 1989.
- [Lus97] G. Lusztig. Cells in affine Weyl groups and tensor categories. *Adv. Math.*, 129(1):85–98, 1997.
- [Lus14] G. Lusztig. Hecke algebras with unequal parameters. [arXiv:math/0208154](https://arxiv.org/abs/math/0208154), 2014.
- [MS89] G. Moore and N. Seiberg. Classical and quantum conformal field theory. *Comm. Math. Phys.*, 123(2):177–254, 1989.
- [Rau12] S. Raum. Isomorphisms and fusion rules of orthogonal free quantum groups and their free complexifications. *Proc. Amer. Math. Soc.*, 140(9):3207–3218, 2012.
- [Tur10] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, revised edition, 2010.
- [Xu] T. Xu. Sage code for Hecke algebras and asymptotic Hecke algebras. <https://github.com/TianyuanXu/mysagecode>.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403
E-mail address: tianyuan@uoregon.edu